Radically Elementary IP Theory Based on Extensive Measurement

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Extensive Structures *A Closed Archimedean Extensive Structure* [**CAES**] provides a (positive) real-valued, scalar representation of a binary relation that is additive in a concatenation operation. We summarize that theory as follows.

Let $\mathscr{D} = \{d_1, d_2, \dots\}$ be a domain of objects. Let $\oplus$ be a function from $\mathscr{D} \times \mathscr{D} \to \mathscr{D}$, understood as a concatenation operation on pair of objects. Finally, let $\succeq$ be a binary relation on $\mathscr{D} \times \mathscr{D}$. Five axioms for a **CAES** are these:

Axiom$_1$ $\succeq$ is a transitive, complete weak order, with symmetric $\approx$ and asymmetric $\succ$ parts.
Axiom$_2$ (Cancellation) $d_1 \succeq d_2$ iff $d_1 \oplus d_3 \succeq d_2 \oplus d_3$.
Axiom$_3$ (Associativity and Commutativity) $d_1 \oplus (d_2 \oplus d_3) \approx (d_2 \oplus d_1) \oplus d_3$.
Axiom$_4$ (Positivity) $d_1 \oplus d_2 \succeq d_1$.

Let $\boldsymbol{n}d = d \oplus d \oplus \cdots \oplus d$ with $\boldsymbol{n} - 1$ concatenations.

Axiom$_5$ (Archimedes) If $d_2 \succ d_1$, and given d_3 and d_4, there exists $\boldsymbol{n}$ such that $[\boldsymbol{n}d_2] \oplus d_3 \succeq [\boldsymbol{n}d_1] \oplus d_4$.

Theorem$_1$ [1] Given a **CAES**, there exists a positive, real-valued function $\boldsymbol{g} : \mathscr{D} \to \Re^+$ where

- $\boldsymbol{g}(d_1) \succeq \boldsymbol{g}(d_2)$ iff $d_1 \succeq d_2$,
- $\boldsymbol{g}(d_1 \oplus d_2) = \boldsymbol{g}(d_1) + \boldsymbol{g}(d_2)$.

and $\boldsymbol{g}$ is unique up to scalars, $\boldsymbol{g}' = \alpha \boldsymbol{g}$ $(\alpha > 0)$.

We call a system that satisfies all but **Axiom$_5$** a *Radically Elementary Closed Extensive Structure* [**RECES**].

Theorem$_2$ [2] Given a **RECES**, there exists a positive, non-standard $^*\Re^+$ valued function $^*\boldsymbol{g} : \mathscr{D} \to {}^*\Re^+$ where

- $^*\boldsymbol{g}(d_1) \geq {}^*\boldsymbol{g}(d_2)$ iff $d_1 \succeq d_2$,
- $^*\boldsymbol{g}(d_1 \oplus d_2) = {}^*\boldsymbol{g}(d_1) + {}^*\boldsymbol{g}(d_2)$.

Regular Probability on a Finite Set as a CAES Let $\Omega = \{\omega_1, \dots, \omega_n\}$ be a finite partition and let $\mathscr{I}$ be a domain of *favorable investments* $\mathscr{I} = \{\boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{I}_3, \dots\}$ where each investment scheme pays a determinate, non-negative dollar return $\mathscr{I}_i(\omega_j) = x_{ij} \geq 0$, as a function of ω. Define concatenation as $\boldsymbol{I}_1 \oplus \boldsymbol{I}_2 = \boldsymbol{I}_3$ where $x_{3j} = x_{1j} + x_{2j}$, $j = 1, \dots, n$. Let $\succeq$ be a binary preference relation between such favorable investment opportunities.

Application$_1$ With a simple modification of **Axiom$_4$** to include the constant $\boldsymbol{I}_0 = \boldsymbol{0}$, by *Theorem*$_1$, if this system is a **CAES** over $\mathscr{I}$, there exists a unique regular probability $\boldsymbol{P}$ on Ω, $\boldsymbol{P}(\omega_j) > 0$, where preference is represented by expected value: $\boldsymbol{g}(\boldsymbol{I}_i) = \sum_j \boldsymbol{P}(\omega_j) x_{ij}$. Let $\emptyset \neq E \subseteq \Omega$. Then, in the usual fashion, $\succeq_E$, called-off preference given E, suffices to define the conditional probability, $\boldsymbol{P}(\cdot|E)$.

Non-Standard Probability on Ω as a RECES

Application$_2$ Drop the Archimedean **Axiom$_5$** from *Application*$_1$ and, by *Theorem*$_2$, preference is a **RECES** that is represented through $^*\boldsymbol{g}$ by a non-standard probability $^*\boldsymbol{P}$ with non-standard expected value, and non-standard, conditional expected value.
Application$_3$ Modify **Axiom$_1$** in *Application*$_1$ so that strict preference is a strict partial order, $\succ$, as in [3, §4 in particular]. A corollary to *Theorem*$_1$ is **IP** theory, where a convex set of probabilities represents $\succ$ and $\succ_E$.
Application$_4$ Continue *Application*$_3$ by dropping **Axiom$_5$**. A corollary to *Theorem*$_2$ is non-standard ***IP** theory, where a convex set of non-standard probabilities and non-standard conditional probabilities represent strict preference and strict called-off preference.
Application$_5$ Continue *Application*$_4$. Replace modified **Axiom$_1$** with **Axioms 1a** and **1b** from [4, p. 164] in the theory of coherent choice functions.

Conjecture This modified **RECES** structure characterizes *all* ***IP** sets of non-standard probabilities on the finite set Ω.

References

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