On the Number and Characterization of the Extreme Points of the Core of Necessity Measures on Finite Spaces

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Abstract

This paper develops a combinatorial description of the extreme points of the core of a necessity measure on a finite space. We use the ingredients of Dempster-Shafer theory to characterize a necessity measure and the extreme points of its core in terms of the Möbius inverse, as well as an interpretation of the elements of the core as obtained through a transfer of probability mass from non-elementary events to singletons. With this understanding we derive an exact formula for the number of extreme points of the core of a necessity measure and obtain a constructive combinatorial insight into how the extreme points are obtained in terms of mass transfers. Our result sharpens the bounds for the number of extreme points given in [15] or [14, 13]. Furthermore, we determine the number of edges of the core of a necessity measure and additionally show how our results could be used to enumerate the extreme points of the core of arbitrary belief functions in a not too inefficient way.

Keywords. necessity measure, core, extreme point, enumeration, belief function, Möbius inverse, mass transfer, possibility measure, credal set, focal set.

1 Introduction

Let $\Omega = \{\omega_1, \dots, \omega_n\}$ be a finite space and let $N : 2^\Omega \longrightarrow [0,1]$ be a necessity measure.[1] The core $\mathscr{M}(N)$ of a necessity measure N is defined as the set of probability measures dominating N:

$$\mathscr{M}(N) := \{P \in \mathscr{P}_n | \forall A \in 2^\Omega : P(A) \geq N(A)\},$$

where $\mathscr{P}_n$ denotes the set of all probability measures[2] on Ω. If one identifies a probability measure P with its characterizing vector $(P(\{\omega_1\}), \dots, P(\{\omega_n\}))$ then the core of N is a convex polytope[3] with finite many extreme points.

[1] A necessity measure $N : 2^\Omega \longrightarrow [0,1]$ is a map satisfying $N(\emptyset) = 0, N(\Omega) = 1$ and $N(A \cap B) = min\{N(A), N(B)\}$ for all $A, B \in 2^\Omega$. For a general introduction to necessity measures, see, e.g., [6].

[2] Since Ω is finite here, it does not make a difference if we take finitely additive or σ-additive probabilities.

[3] For basics of polytopes, see, e.g., [12].

The aim of this paper is to give a formula for the number as well as a constructive description of these extreme points. Since we will derive an exact formula for the number of extreme points in this paper, we are in fact able to improve the bounds for the number of extreme points given in [14, 13] that are not tight.

Studying the geometry of the core and describing the extreme points of the core is interesting for its own, not only in the context of necessity measures. Furthermore, for different applications of imprecise probability theory it is helpful to efficiently describe and compute the extreme points of the core to make different computational tasks tractable.

For example in decision making under partial prior information, for one approach for computing optimal decisions given in [19, Section 4], one needs to compute all extreme points of the underlying imprecise probability model. Also for statistical hypothesis testing under imprecise probabilistic models one can use the extreme points of the cores of the underlying models for the construction of Niveau-α-Maximin-Tests tests, cf., [1, Section 4,5].

In the field of game theory, where more general set functions (games) are treated, the core is an object of interest as well, cf., e.g. [10, 18, 3]. There, for example in the context of convex games the so-called Shapley value appears as the center of gravity of the extreme points of the core (cf., [18]).

The idea of studying complex set functions (here, necessity measures, or more generally, belief functions) via a characterizing set of more easy to handle set functions (here, classical probability measures) is also present in the context of qualitative capacities (cf., e.g., [11]), where the so-called possibilistic core consisting of all (qualitative) possibility measures dominating a given qualitative capacity was introduced in [7]. There, results similar to Theorem 2 of our paper and an enumerating procedure for the “extreme points” of this possibilistic core (which are defined differently in an order theoretic manner) are given.

To describe the core of necessity measures we use

Dempster-Shafer theory[4] and treat necessity measures as special kinds of belief functions. A belief function $\text{Bel} : 2^\Omega \longrightarrow [0,1]$ is a function that is induced by a so-called basic probability assignment $m : 2^\Omega \longrightarrow [0,1]$ via

$$\forall A \in 2^\Omega : \text{Bel}(A) = \sum_{B \subseteq A} m(A).$$

The basic probability assignment m generating *Bel* can be interpreted as a generalization of a probability measure that assigns probability mass not only to elementary events but also to any arbitrary event in $2^\Omega \backslash \{\emptyset\}$. Since m is thought of as a probability measure, it is assumed that $\sum_{A \in 2^\Omega} m(A) = 1$ and furthermore $m(\emptyset) = 0$. Events $A \subseteq \Omega$ with $m(A) > 0$ are called focal sets and the set of all focal sets of a belief function Bel is denoted with $\mathscr{F}(\text{Bel})$. The motivation for introducing the basic probability assignment is the modeling of some kind of uncertainty that cannot be associated with exactly one state $\omega \in \Omega$, but only with a non-elementary event $A \subseteq \Omega$. The belief function *Bel* induced by the basic probability assignment is then of interest if one wants to know for some set A, which portion of the whole probability mass can overall be associated to the states of A. For a given belief function Bel the basic probability assignment m generating Bel can be recovered from Bel by applying the so-called Möbius inversion, thus m is also called the Möbius inverse of Bel.

Now, a necessity measure N (on a finite space) can be characterized[5] as a special belief function[6] where all focal sets are nested, i.e.: $\forall A, B \in \mathscr{F}(N) : A \subseteq B$ or $B \subseteq A$. The core of a belief function can be understood as the set of all probability measures that are consistent with the belief function in the sense that every $P \in \mathscr{M}(Bel)$ can be obtained via a "transfer" of probability mass of the basic probability assignment m from non-elementary events $A \subseteq \Omega$ to singletons $\{\omega\} \subseteq A$. To make this more precise, we state the following definition and theorem:

Definition 1 *Let Bel be a belief function with corresponding basic probability assignment m. A selection* $\lambda : \mathscr{F}(\text{Bel}) \longrightarrow \mathscr{P}_n : A \mapsto \lambda_A$ *is a mapping that assigns to every focal set A a probability measure* λ_A *whose support is in A. The set of all selections associated to a belief function on a space* 2^Ω *with* $|\Omega| = n$ *is denoted with* Λ_n. *A selection* λ *could be understood as specifying for every focal set A and for every state* $\omega \in A$, *how much of mass assigned to A should be transferred from A to* ω. *More precisely, for a belief function* Bel *and a selection* λ *there is an induced probability measure* P_λ *via*

$$P_\lambda(\{\omega_i\}) = \sum_{A \in \mathscr{F}(\text{Bel})} m(A) \cdot \lambda_A(\{\omega_i\}).$$

[4]For an introduction, see, e.g., [17].

[5]For a proof, see, e.g., [17, p.220].

[6]Note that the interpretation of a necessity measure is not necessarily identical to that of a belief function, in this paper we analyze only purely mathematical properties of necessity measures in the framework of Dempster-Shafer theory.

Theorem 1 *For a belief function Bel we have*

$$\mathscr{M}(Bel) = \{P_\lambda \mid \lambda \in \Lambda_n\}.$$

The proof can be found in [4, Corollary 3, p.273] or, in the context of game theory, in [5, Theorem 2]. In the context of game theory, the set $\{P_\lambda \mid \lambda \in \Lambda_n\}$ is called selectope and the set $\mathscr{M}(\nu)$, where ν is a game, is called core and both sets coincide iff the Möbius inverse of the game ν is non-negative, as is also shown in [5, Theorem 2]. Since selections are simply mappings, we can introduce convex combinations. For selections $\lambda, \lambda' \in \Lambda_n$ and $c \in [0,1]$ define

$$c \cdot \lambda + (1-c) \cdot \lambda' : \mathscr{F}(\text{Bel}) \longrightarrow \mathscr{P}_n : \\ A \mapsto c \cdot \lambda_A + (1-c) \cdot \lambda'_A.$$

Note that the probability measure associated to a convex combination of two selections equals the convex combination of the probability measures associated to the two selections: For $\lambda, \lambda' \in \Lambda_n$ and $c \in [0,1]$ we have

$$P_{c\lambda + (1-c)\lambda'} = cP_\lambda + (1-c)P_{\lambda'}.$$

This suggests that it is possible to characterize the extreme points of $\mathscr{M}(Bel)$ in terms of the corresponding selections in Λ_n.

Lemma 1 *For an extreme point* $P = P_\lambda \in \mathscr{M}(Bel)$ *we have:* $\forall A \in \mathscr{F}(Bel) : \exists! \omega \in A : \lambda_A(\{\omega\}) = 1$.

Proof: Let $A \in \mathscr{F}(Bel)$. If for all $\omega \in A : \lambda_A(\{\omega\}) \neq 1$ then there would exist $\omega_i, \omega_j \in A$ with $\lambda_A(\{\omega_i\}) > 0$ and $\lambda_A(\{\omega_j\}) > 0$. Now set $\varepsilon := \min\{\lambda_A(\{\omega_i\}), \lambda_A(\{\omega_j\})\} > 0$ and define the selections μ and ν via

$$\mu_B(\{\omega\}) = \begin{cases} \lambda_B(\{\omega\}) & \text{if } B \neq A \\ \lambda_A(\{\omega\}) & \text{if } B = A, \omega \notin \{\omega_i, \omega_j\} \\ \lambda_A(\{\omega\}) + \varepsilon & \text{if } B = A, \omega = \omega_i \\ \lambda_A(\{\omega\}) - \varepsilon & \text{if } B = A, \omega = \omega_j \end{cases};$$

$$\nu_B(\{\omega\}) = \begin{cases} \lambda_B(\{\omega\}) & \text{if } B \neq A \\ \lambda_A(\{\omega\}) & \text{if } B = A, \omega \notin \{\omega_i, \omega_j\} \\ \lambda_A(\{\omega\}) - \varepsilon & \text{if } B = A, \omega = \omega_i \\ \lambda_A(\{\omega\}) + \varepsilon & \text{if } B = A, \omega = \omega_j \end{cases}.$$

Then $P_\lambda = \frac{1}{2}P_\mu + \frac{1}{2}P_\nu$ and $P_\mu \neq P_\nu$ because

$$\begin{aligned} P_\mu(\{\omega_i\}) - P_\nu(\{\omega_i\}) &= \sum_{B \neq A} m(B) \cdot \mu_B(\{\omega_i\}) + m(A) \cdot \mu_A(\{\omega_i\}) \\ &\quad - \sum_{B \neq A} m(B) \cdot \nu_B(\{\omega_i\}) - m(A) \cdot \nu_A(\{\omega_i\}) \\ &= 2\varepsilon \cdot m(A) \neq 0. \end{aligned}$$

This is a contradiction to the assumption that P_λ is an extreme point of $\mathscr{M}(\mathrm{Bel})$, so there exists an ω with $\lambda_A(\{\omega\}) = 1$. Because λ_A is a probability measure, there could be only one ω with $\lambda_A(\omega) = 1$. ∎

Lemma 1 suggests the following definition:

Definition 2 *Let $\mathscr{D}_n := \{P \in \mathscr{P}_n \mid \exists!\omega \in \Omega : P(\{\omega\}) = 1\}$ and let* Bel *be a belief function. Let furthermore λ be a selection and $A \in \mathscr{F}(\mathrm{Bel})$. If $\lambda_A \in \mathscr{D}_n$ we denote by $\omega_\lambda(A)$ the unique ω with $\lambda_A(\{\omega\}) = 1$.*

Theorem 2 *Let* Bel *be a belief function and let P_λ be an extreme point of the core of* Bel*. For focal sets $A, A' \in \mathscr{F}(\mathrm{Bel})$ with $\{\omega_\lambda(A), \omega_\lambda(A')\} \subseteq A \cap A'$ we have*

$$\omega_\lambda(A) = \omega_\lambda(A').$$

Proof: Assume that $\omega_\lambda(A) \neq \omega_\lambda(A')$. We now show that if this would be the case then we could construct two different elements P_μ and P_ν of the core of Bel such that $P_\lambda = cP_\mu + (1-c)P_\nu$ for some appropriate chosen $c \in [0,1]$ and thus P_λ could not be an extreme point, so $\omega_\lambda(A) = \omega_\lambda(A')$: Define the selections μ and ν as

$$\mu_B(\omega) = \begin{cases} \lambda_B(\omega) & \text{if } B \neq A' \\ 1 & \text{if } B = A', \omega = \omega_\lambda(A) \\ 0 & \text{else} \end{cases}$$

$$\nu_B(\omega) = \begin{cases} \lambda_B(\omega) & \text{if } B \neq A \\ 1 & \text{if } B = A, \omega = \omega_\lambda(A') \\ 0 & \text{else} \end{cases}.$$

These selections lead in fact to two different probability measures P_μ and P_ν. Now, with $c = \frac{m(A)}{m(A)+m(A')}$ we have $P^* := c \cdot P_\mu + (1-c) \cdot P_\nu = P_\lambda$. To see this, look at the three different cases $\omega = \omega_\lambda(A), \omega = \omega_\lambda(A')$ and $\omega \notin \{\omega_\lambda(A), \omega_\lambda(A')\}$:

$$\begin{aligned} P^*(\{\omega_\lambda(A)\}) &= c \sum_{\substack{B \neq A', \\ \omega_\lambda(B)=\omega_\lambda(A)}} m(B) + m(A') + (1-c) \sum_{\substack{B \neq A, \\ \omega_\lambda(B)=\omega_\lambda(A)}} m(B) \\ &= \sum_{\substack{B \notin \{A,A'\} \\ \omega_\lambda(B)=\omega_\lambda(A)}} m(B) + c \cdot (m(A) + m(A')) \\ &= \sum_{\substack{B \notin \{A,A'\} \\ \omega_\lambda(B)=\omega_\lambda(A)}} m(B) + m(A) \\ &= P_\lambda(\{\omega_\lambda(A)\}). \end{aligned}$$

Here, the first sum in the first equation is valid because of Lemma 1 and because all mass $m(A')$ is assigned by μ to $\omega_\lambda(A)$ and the second sum does not contain $m(A)$ and $m(A')$ because the mass $m(A)$ and $m(A')$ is assigned by ν to $\omega_\lambda(A') \neq \omega_\lambda(A)$.

$$\begin{aligned} P^*(\{\omega_\lambda(A')\}) &= c \sum_{\substack{B \neq A', \\ \omega_\lambda(B)=\omega_\lambda(A')}} m(B) + (1-c) \sum_{\substack{B \neq A, \\ \omega_\lambda(B)=\omega_\lambda(A')}} m(B) + m(A) \\ &= \sum_{\substack{B \notin \{A,A'\} \\ \omega_\lambda(B)=\omega_\lambda(A')}} m(B) + (1-c)(m(A') + m(A)) \\ &= \sum_{\substack{B \notin \{A,A'\} \\ \omega_\lambda(B)=\omega_\lambda(A')}} m(B) + m(A') \\ &= P_\lambda(\{\omega_\lambda(A')\}). \end{aligned}$$

Analogously, here, the first sum in the first equation does not contain $m(A)$ and $m(A')$ because these masses are assigned by μ to $\omega_\lambda(A) \neq \omega_\lambda(A')$ and in the second sum the mass $m(A)$ is assigned by ν to $\omega_\lambda(A')$. For $\omega \notin \{\omega_\lambda(A), \omega_\lambda(A')\}$ we have

$$\begin{aligned} P^*(\{\omega\}) &= c \sum_{\substack{B \neq A' \\ \omega_\lambda(B)=\omega}} m(B) + (1-c) \sum_{\substack{B \neq A \\ \omega_\lambda(B)=\omega}} m(B) \\ &= \sum_{\substack{B \notin \{A,A'\} \\ \omega_\lambda(B)=\omega}} m(B) = P_\lambda(\{\omega\}). \end{aligned}$$

Here, the masses $m(A)$ and $m(A')$ essentially play no role, because they are not assigned to ω by neither μ nor ν. ∎

2 Description of the Core of a Necessity Measure

Now we are prepared to describe the extreme points of the core of a necessity measure. As already mentioned, a necessity measure N is a belief function where the focal sets are nested. This enables a concise description of the extreme points of the core:

Theorem 3 *Let N be a necessity measure with focal sets $\mathscr{F}(N) = \{A_1 \subset A_2 \subset \ldots \subset A_k\}$. The number of extreme points of the core $\mathscr{M}(N)$ is given by*

$$|\mathrm{ext}(\mathscr{M}(N))| = |A_1| \cdot \prod_{i=2}^{k} (|A_i \backslash A_{i-1}| + 1). \qquad (1)$$

Furthermore, the set of extreme points can be described as

$$\mathrm{ext}(\mathscr{M}(N)) = \{P_\lambda \mid \lambda \in \Lambda_n^{\mathrm{ext}}\}$$

with $\Lambda_n^{\mathrm{ext}} = \{\lambda \in \Lambda_n \mid \forall A_l \in \mathscr{F}(N) : \lambda_{A_l} \in \mathscr{D}_n \;\&\; \omega_\lambda(A_l) \in A_{l-1} \Rightarrow \omega_\lambda(A_l) = \omega_\lambda(A_{l-1})\}$.

Proof: We firstly show that the number of extreme points is lower or equal to $|A_1| \cdot \prod_{i=2}^{k}(|A_i \backslash A_{i-1}| + 1)$. For this we only have to observe that we could inductively look at the focal sets of N starting from the smallest focal set A_1. For a given extreme point P_λ, the mass assigned to A_1 can be assigned to any $\omega \in A_1$, for which one has $|A_1|$ possibilities. Then, for the second focal set A_2 one has $|A_2 \backslash A_1|$ possibilities to assign the mass of A_2 outside of A_1 and only one possibility to assign the mass into A_1 because in this case, the element $\omega \in A_1$, to which the mass is assigned, is, because of Theorem 2, already determined as $\omega = \omega_\lambda(A_1)$, so for the assignment of the mass $m(A_2)$, we have maximal $|A_i \backslash A_{i-1}| + 1$ possibilities and so on. This gives maximal $|A_1| \cdot \prod_{i=2}^{k}(|A_i \backslash A_{i-1}| + 1)$ possibilities for constructing an extreme point.

Now we still have to show that the extreme points constructed in the above manner are all actually extreme points and that they are all pairwise different. For this, we can analogously look at ascending focal elements. To see that any P_λ with $\lambda \in \Lambda_n^{\text{ext}}$ is in fact an extreme point we firstly assume that P_λ is the convex combination of r extreme points P_{μ_i} with μ_i in Λ_n^{ext} and show that then necessarily $P_\lambda = P_{\mu_1} = \ldots = P_{\mu_r}$ which shows that P_λ is an extreme point of $\mathscr{M}(N)$:

Since P_λ is such that $\lambda \in \Lambda_n^{\text{ext}}$, all mass of A_1 is assigned by λ to exactly one $\omega \in A_1$ and no other mass $m(B)$ is assigned by λ to some other $\omega \in A_1$, so $P_\lambda(A_1 \backslash \{\omega_\lambda(A_1)\}) = 0$. This implies that for all P_{μ_i} we also have $P_{\mu_i}(A_1 \backslash \{\omega_\lambda(A_1)\}) = 0$ and so $\lambda(A_1) = \mu_1(A_1) = \ldots = \mu_r(A_1)$. Now, look at A_2. If λ assigns the mass of A_2 somewhere into A_1 (namely to $\omega_\lambda(A_1)$), then no mass at all is assigned by λ to some $\omega \in A_2 \backslash A_1$ and thus necessarily all μ_i also have to assign all the mass into A_1 (namely to $\omega_\lambda(A_1)$), so, in this cases we have $\lambda(A_2) = \mu_1(A2) = \ldots = \mu_r(A_2)$. If λ assigns all mass of A_2 somewhere into $A_2 \backslash A_1$, then every μ_i also has to assign the mass of A_2 outside A_1 because if there was a P_{μ_i} that assigns the mass of A_2 into A_1 then we would have $P_{\mu_i}(A_1) > Bel(A_1) = P_\lambda(A_1)$ because if λ assigns the mass of A_2 not into A_1, then λ assigns also the mass of all further $A_3, \ldots, A_k$ not into A_1 and thus $P_\lambda(A_1) = Bel(A_1)$. But if $P_{\mu_i}(A_1) > P_\lambda(A_1)$ then because P_λ is assumed to be a convex combination of $P_{\mu_1}, \ldots, P_{\mu_r}$, there has to be a P_{μ_j} with $P_{\mu_j} < P_\lambda(A_1) = Bel(A_1)$. This is a contradiction to the fact that P_{μ_j} dominates Bel. So, in fact, in this case all μ's assign the mass of A_2 outside of A_1 and thus exactly to $\omega_\lambda(A_2)$ because $P_\lambda(A_2 \backslash \{\omega_\lambda(A_2)\}) = 0$. The same argumentation for all further $A_3, \ldots, A_k$ shows that altogether $\lambda(A_l) = \mu_1(A_l) = \ldots = \mu_r(A_l)$ for $l = 1, \ldots, k$ and so $P_\lambda = P_{\mu_1} = \ldots = P_{\mu_r}$.

To finally see that selections λ, λ' with at least one focal set A_l with $\omega_\lambda(A_l) \neq \omega_{\lambda'}(A_l)$ lead to different P_λ and $P_{\lambda'}$ look at the smallest focal set A_l with $\omega_\lambda(A_l) \neq \omega_{\lambda'}(A_l)$. If $l = 1$ then $P_\lambda(\{\omega_\lambda(A_l)\}) > 0$ and $P_{\lambda'}(\{\omega_\lambda(A_l)\}) = 0$ so P_λ and $P_{\lambda'}$ are different. If $l > 1$ then we have $\omega_\lambda(A_l) \notin A_{l-1}$ or $\omega_{\lambda'}A(_l) \notin A_{l-1}$ because if both $\omega_\lambda(A_l)$ and $\omega_{\lambda'}(A_l)$ were in A_{l-1} then also $\omega_\lambda(A_{l-1})$ and $\omega_{\lambda'}(A_{l-1})$ would differ which would be a contradiction to the minimality of l. So assume without loss of generality $\omega_\lambda(A_l) \notin A_{l-1}$. Then $P_\lambda(\{\omega_\lambda(A_l)\}) > 0$ but $P_{\lambda'}(\{\omega_\lambda(A_l)\}) = 0$ because λ' assigns all mass of focal sets $A \supseteq A_l$ either outside of A_l or to $\omega_{\lambda'}(A_l)$ and all other focal sets $A \subseteq A_{l-1}$ do not contain $\omega_\lambda(A_l)$. ■

With Theorem 3 we have a precise constructive description of the extreme points of the core of a necessity measure. It turns out that it is possible to give furthermore a formula for the number of edges of the core. For this purpose we can use the fact that if two extreme points P and P' are connected through an edge of the core, then they differ exactly at two states and thus the difference of P and P' is of the form $P - P' = (0, \ldots, 0, \varepsilon, 0, \ldots, -\varepsilon, 0 \ldots, 0)$ for some $\varepsilon \in \mathbb{R}$. This result is given in [20] that more generally treats capacities of order 2.

Definition 3 *Let* Bel *be a belief function with focal elements* $\mathscr{F}(\text{Bel}) = \{A_1, \ldots, A_k\}$ *and let* P_λ *be an extreme point of the core of* Bel *induced by a selection* λ. *The characteristic* χ *of* λ *is defined*[7] *as*

$$\chi : \mathscr{F}(Bel) \to \Omega \,\dot{\cup}\, \{0\} :$$
$$A_i \mapsto \begin{cases} 0 & \textit{if } \exists j < i : \omega_\lambda(A_j) = \omega_\lambda(A_i) \\ \omega_\lambda(A_i) & \textit{else} \end{cases}.$$

Lemma 2 *Let N be a necessity measure with focal sets* $\mathscr{F}(N) = \{A_1 \subset A_2 \subset \ldots \subset A_k\}$ *and* P_λ *and* $P_{\lambda'}$ *two different extreme points of* $\mathscr{M}(N)$ *induced by selections* λ *and* λ' *with corresponding characteristics* χ *and* χ'. *Then* P_λ *and* $P_{\lambda'}$ *are adjacent (meaning connected through an edge of* $\mathscr{M}(N)$*) if and only if there is exactly one focal set A with* $\chi(A) \neq \chi'(A)$.

Proof: Assume that P_λ and $P_{\lambda'}$ are adjacent and that there are two different focal sets where the characteristics χ and χ' differ. Look particularly at the smallest set A_l and some other set A_r where χ and χ' differ. Then P_λ and $P_{\lambda'}$ differ at the two different states $\omega_\lambda(A_l)$ and $\omega_{\lambda'}(A_l)$. Since furthermore either $\omega_\lambda(A_r)$ or $\omega_{\lambda'}(A_r)$ is not in A_l there exists a third state $\omega_\lambda(A_r)$ or $\omega_{\lambda'}(A_r)$ where P_λ and $P_{\lambda'}$ differ, so P_λ and $P_{\lambda'}$ could not be adjacent. This shows that in fact adjacent extreme points have characteristics that differ only on one focal set.

Let now λ and λ' be two selections with associated characteristics χ and χ' that differ only on one focal set A_l. For arbitrary $\omega \in \Omega$ let $i(\omega)$ denote the index of the smallest focal set that contains ω. Then for $\omega \notin \{\omega_\lambda(A_l), \omega_{\lambda'}(A_l)\}$ we have that $P_\lambda(\{\omega\}) = 0$ if $\chi(A_{i(\omega)}) \neq \omega$ and otherwise if $\chi(A_{i(\omega)}) = \omega$ that

$$P_\lambda(\{\omega\}) = \begin{cases} m(A_{i(\omega)}) + \sum_{\substack{B \in \{A_{i(\omega)}, \ldots, A_k\} \\ \chi(B)=0}} m(B) & \text{if } i(\omega) > l, \\ m(A_{i(\omega)}) + \sum_{\substack{B \in \{A_{i(\omega)}, \ldots, A_{l-1}\} \\ \chi(B)=0}} m(B) & \text{if } i(\omega) < l. \end{cases}$$

So $P_\lambda(\{\omega\}) = P_{\lambda'}(\{\omega\})$. This means that P_λ and $P_{\lambda'}$ differ at most at two states (namely $\omega_\lambda(A_l)$ and $\omega'_\lambda(A_l)$) and since they are different, they differ exactly at two states. Unfortunately, extreme points that differ only at two states need not to be adjacent (see for example the belief function of section 4) but in the case of necessity measures this is the case. In fact, one can show with the concepts of [20] that for extreme points with associated characteristics that differ only at one focal set (or equivalently, for extreme points that differ only at two states,) there exist permutations σ and μ such that $P_\lambda = p_\sigma, P_{\lambda'} = p_\mu$ and the associated equivalence classes $[p_\sigma]$ and $[p_\mu]$ are neighboured in the network and thus P_λ and $P_{\lambda'}$ are adjacent. Details about this can be given upon request. ■

From Lemma 2 it follows that every extreme point P_λ has $|A_1| - 1 + \sum_{i=1}^k (|A_i \backslash A_{i-1}|)$ adjacent extreme points. With this we can count the number of edges of $\mathscr{M}(N)$:

[7] Note that this definition depends on the numbering of the focal sets. For the special case of necessity measures the focal sets are assumed to be numbered in increasing cardinality.

Theorem 4 *Let N be a necessity measure with focal sets $\mathscr{F}(N) = \{A_1 \subset A_2 \subset \ldots \subset A_k\}$. The number $|\text{edges}(\mathscr{M}(N))|$ of edges of the core $\mathscr{M}(N)$ is given by*

$$\frac{1}{2} \cdot |A_1| \cdot \prod_{i=2}^{k} (|A_i \backslash A_{i-1}| + 1) \cdot (|A_1| - 1 + \sum_{i=2}^{k} |A_i \backslash A_{i-1}|).$$

Proof: The statement about the number of edges follows simply by counting for all extreme points P_λ all adjacent extreme points $P_{\lambda'}$ that form an edge with P_λ and by taking into account that with this, every edge is counted two times. ■

We now compare our result with results given in [14, 13]. There, the results are given in the language of possibility measures Π that are defined in a dual way as $\Pi : 2^\Omega \longrightarrow [0,1] : A \mapsto 1 - N(A^c)$ and are then join preserving mappings particularly satisfying $\Pi(A) = \max_{\omega \in A} \Pi(\{\omega\})$ and are thus uniquely defined through $\pi_i := \Pi(\{\omega_i\})$. Furthermore, in the sequel we assume $0 < \pi_1 \leq \pi_2 \leq \ldots \leq \pi_n = 1$ to simplify presentation. In [14, 13] the set $S := \{i \in \{1,\ldots,n-2\} \mid \pi_{i+1} > \pi_i\} \cup \{n-1\}$ and its cardinality $s := |S|$ play an important role in establishing bounds for the number of extreme points. In terms of the necessity measure N the set S writes as $S = \{i \in \{1,\ldots,n-2\} \mid \{\omega_{i+1},\ldots,n\} \in \mathscr{F}(N)\} \cup \{n-1\}$ and s equals the number of non-elementary focal sets.

Theorem 5 ([14, 13]) [8] *Let N be a necessity measure with associated possibility measure Π satisfying $0 < \pi_1 \leq \ldots \leq \pi_n = 1$. Let s denote the number of non-elementary focal sets of N (or equivalently the cardinality of the set $S = \{i \in \{1,\ldots,n-2\} \mid \pi_{i+1} > \pi_i\} \cup \{n-1\}$). Then the core $\mathscr{M}(N)$ is a $n-1$ dimensional simple polytope*[9] *with $n-1+s$ facets. The number of extreme points is bounded by*

$$|\text{ext}(\mathscr{M}(N))| \geq s(n-2)+2, \tag{2}$$

$$|\text{ext}(\mathscr{M}(N))| \leq \binom{n-2+s-\lfloor\frac{n-2}{2}\rfloor}{\lfloor\frac{n-1}{2}\rfloor} + \binom{n-2+s-\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{n-2}{2}\rfloor} \tag{3}$$

and by

$$|\text{ext}(\mathscr{M}(N)| \leq 2^s \prod_{j=1}^{s} (i_j - i_{j-1}) \tag{4}$$

where $i_0 = 0$ and $i_1, i_2, \ldots, i_s$ denote the increasingly ordered indices of the set S.

3 Illustration of the Results

We can now illustrate our results via an example taken from [14, Example 2, p.242]. There, $\Omega = \{\omega_1, \ldots, \omega_5\}$ and a possibility measure Π is given by $\pi_1 = 0$, $\pi_2 = \pi_3 = 0.5$, $\pi_4 = 0.75$, $\pi_5 = 1$. This translates to an associated necessity measure N with focal elements $A_1 = \{\omega_5\}$, $A_2 = \{\omega_4, \omega_5\}$ and $A_3 = \{\omega_2, \omega_3, \omega_4, \omega_5\}$ and masses $m(A_1) = 0.25$, $m(A_2) = 0.25$, $m(A_3) = 0.5$. Because of $\pi_1 = 0$ we have $P(\{\omega_1\}) = 0$ for all $P \in \mathscr{M}(N)$ and thus state ω_1 plays essentially no role and the core $\mathscr{M}(N)$ is a 3-dimensional polytope that is uniquely described by the second, third and fourth component of all probability vectors p of the core.

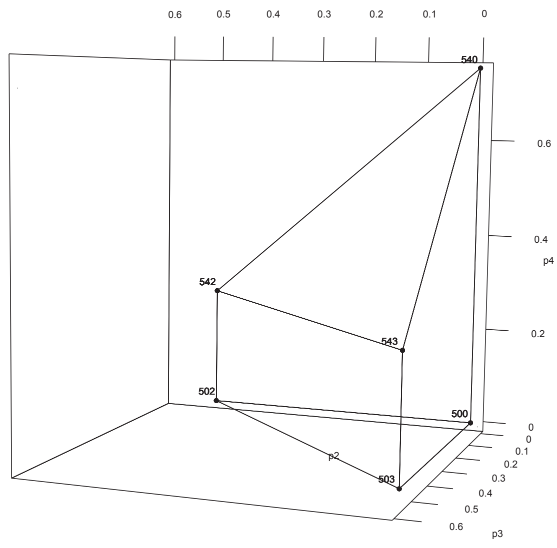

Figure 1: Illustration of the core of Example 2 given in [14].

Figure 1 depicts the core of N. One can see its 6 extreme points, its 9 edges and its 5 facets. This is in accordance with Theorem 3, Theorem 4 and Theorem 5:

$$|\text{ext}(\mathscr{M}(N))| = 1 \cdot 2 \cdot 3 = 6$$
$$|\text{edges}(\mathscr{M}(N))| = \frac{1}{2} \cdot 1 \cdot 2 \cdot 3 \cdot (0+1+2) = 9$$
$$|\text{fac}(\mathscr{M}(N))| = 5 - 2 + 2 = 5.$$

Furthermore, exactly $0+1+2 = 3$ edges meet at every extreme point as argued in the leader of Theorem 4. The digit sequence at every extreme point in Figure 1 indicates the characteristic of the corresponding selection. For example the sequence 503 at the extreme point in the foreground means that the mass of A_1 is assigned to ω_5, the mass of A_2 is assigned to the same ω as the mass of A_1 (thus to ω_5) and the mass of A_3 is assigned to ω_3. One can see that the characteristics of two different extreme points differ exactly at one position if and only if they are adjacent.

The extreme point with characteristic 500 is in a sense distinguished because it is obtained as all mass is assigned to one state ω_5. For every arbitrary necessity measure there exists (at least) one such degenerate extreme point $p \in \mathscr{D}_n$, namely $p = (0, \ldots, 1)$. (If the smallest focal set contains k

[8] Note that unfortunately the bounds given in [14, Theorem 2] are misprinted, the correct bounds can be found in [13].

[9] A d-dimensional polytope is called simple, if all vertices are contained in exactly d facets.

$n-1$	s	2^{n-1}	l_1	u_1	u_2	l_3	u_3
2	2	4	4	4	4	4	4
2	1	4	3	3	4	3	3
3	3	8	8	8	8	8	8
3	2	8	6	6	6	6	6
3	1	8	4	4	4	4	4
4	4	16	14	20	16	16	16
4	3	16	11	14	16	12	12
4	2	16	8	9	16	8	9
8	8	256	58	660	256	256	256
8	7	256	51	450	256	192	192
8	6	256	44	294	256	128	144
8	5	256	37	182	256	80	108
9	9	512	74	1430	512	512	512
9	8	512	66	990	512	384	384
9	7	512	58	660	512	256	288
9	6	512	50	420	512	160	216
10	5	1024	47	378	1024	112	243
15	5	32768	72	1584	7776	192	1024
20	5	1048576	97	5005	32768	272	3125
20	10	1048576	192	277134	1048576	6144	59049
$m \cdot s$	s	$2^{m \cdot s}$	$s(m \cdot s-1)+2$	$\binom{(m+1)s+1-\lfloor \frac{m \cdot s+1}{2} \rfloor}{\lfloor \frac{m \cdot s}{2} \rfloor}+\binom{(m+1)s+1-\lfloor \frac{m \cdot s}{2} \rfloor}{\lfloor \frac{m \cdot s+1}{2} \rfloor}$	$2^s \cdot m^s$	$2^{s-1} \cdot ((m-1)s+2)$	$(m+1)^s$

Table 1: Different bounds for the number of extreme points of the core of a necessity measure for different sizes of $n-1$ and s.

elements then there are even k degenerate extreme points). This extreme point p is adjacent to extreme points of the form $(0,\dots,\pi_k,\dots,1-\pi_k)$ obtained bay assigning all mass $m(A)$ to ω_k if $\omega_k \in A$ and to ω_n else.

Additionally, we can investigate the behaviour of the different bounds for the number of extreme points for different sizes of $n-1$ and s. Table 1 shows the exponential bound 2^{n-1} given in [15], the lower bound l_1 and the upper bound u_1 of [14] (these are here the inequalities (2) and (3)) and the upper bound u_2 of [13] (here inequality (4)) obtained by maximizing (4) under fixed sizes of $n-1$ and s). Additionally, the herein established bounds l_3 and u_3 obtained via minimizing/maximizing (1) for fixed $n-1$ and s are given in the last columns. The last row shows the general situation when $n-1$ is a multiple of s. The sharp upper bound u_3 is obtained by choosing s focal sets $A_1,\dots,A_s$ with cardinality $|A_l| = l \cdot m+1$ where $m=(n-1)/s$. One can see that for fixed m this bound is exponential in s and in the special case of $m=1$ we get the bound $2^s = 2^{1 \cdot s} = 2^{n-1}$ of [15]. For higher m the expansion rate of the exponential growth of the extreme points in dependence on s is greater. If the "density" $\frac{1}{m}$ of focal sets decreases and n is fixed, then the number $(m+1)^s = (m+1)^{\frac{n-1}{m}}$ decreases. For a fixed number of focal sets the number of extreme points is polynomial in the reciprocal m of the density of focal elements.

Our result on the description of the extreme points suggests that it is possible to enumerate all extreme points in a time proportional to $(m+1)^s \cdot s$ because for every extreme point one needs to add s mass values $m(A)$ to some state $\omega^* \in A$ as $p(\omega^*) = p(\omega^*) + m(A)$ to obtain this extreme point.

To get an impression about the possible gain in efficiency, we compare the term $(m+1)^s \cdot s$ with the time two standard enumeration procedures need to enumerate the extreme points. We used implementations of firstly the Double Description Method (cf., [8, 16]) and secondly the Reverse Search Method (cf., [2]) to enumerate the extreme points for different values of m and s and necessity measures that maximize the number of extreme points for given values of m and s.

Figure 2 shows the logarithm of the execution time[10] t in seconds in dependence of s (or m respectively) for the Double Description Method[11] where the value of m (or s respectively) was fixed at different levels. The term $\ln((m+1)^s \cdot s) = s\ln(m+1) + \ln(s)$ is approximately linearly increasing in s (with slope roughly $\ln(m+1)$) and logarithmically increasing in m. Compared to this, the log of computation time increases seemingly linearly in s, but with higher slopes. For example for $m=7$ the slope of $\ln(t)$ is around 4 whereas the slope of $\ln((m+1)^s \cdot s)$ is

[10] We used a personal computer (64 bit) with an Intel(R) Xeon(R) CPU (E5-2650v2, 2.60 Ghz, 2 cores).

[11] We used the r-package rcdd (cf., [9]) which is an interface to the C++ implementation [8] of the Double Description Method.

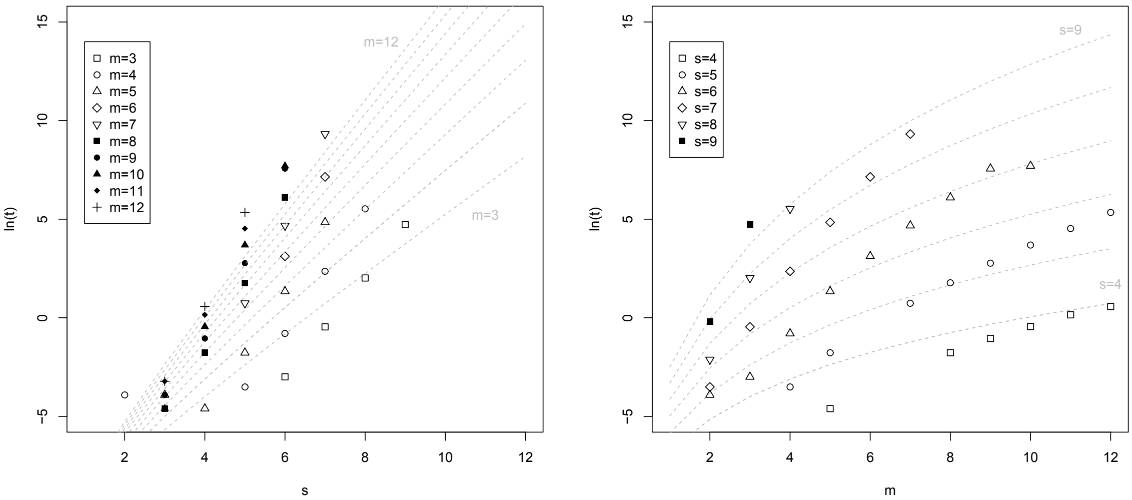

Figure 2: Different execution times of the Double Description Method together with the logarithm of a multiple of the term $(m+1)^s \cdot s$ (grey dashed lines) expected for an efficient enumerating procedure that uses our result.

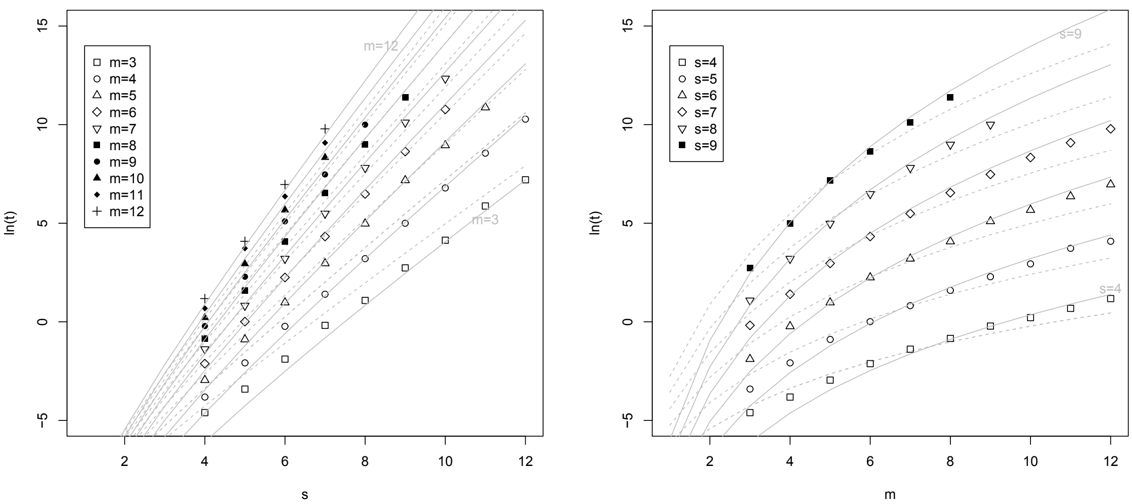

Figure 3: Different execution times of the Reverse Search Method together with the logarithm of a multiple of the term $(m+1)^s \cdot s$ (grey dashed lines) and the logarithm of a multiple of the term $(m+1)^s \cdot (ms)^2$ (grey lines).

somewhere around 2.3, so the expansion rate of the seemingly exponential growth of computation time is larger than the computation time expected for an ideal enumeration procedure. Also the growing of $\ln(t)$ in dependence on m seems to be linearly, so computation time seems to grow also exponentially in m and not polynomially as would be the case with an ideal enumeration procedure.

Figure 3 shows the results for the Reverse Search Method.[12] For this method it is known (cf., [2, Theorem 6.2]) that for simple polytopes the time complexity for enumerating the extreme points is $O(kn)$ per vertex, where n is the number of variables and k is the number of inequalities in the H-representation of the polytope. This would translate in our case to a time complexity of $O((m+1)^s \cdot (ms)^2)$ since we have $(m+1)^s$ vertices of a polytope of dimension $n-1 = m \cdot s$ that could be described by $O(n-1)$ inequalities (cf., [14, p.238]).

It turns out that the execution times are mostly smaller for the Reverse Search Method compared to the Double Description Method. In Figure 3 the grey dashed lines again display the logarithm of a multiple of the term $(m+1)^s \cdot s$, whereas the grey solid lines show the logarithm of a multiple of the term $(m+1)^s \cdot (ms)^2$. One can see that the theoretical time complexity of the Reverse Search Method is roughly in accordance with the actually obtained execution times and that one could still gain some improvement of performance if one uses our results to enumerate the extreme points instead of using the Reverse Search Method.

4 Extension to Belief Functions

With the insight of Theorem 3 and its proof we have not only an exact formula for the number of the extreme points of the core of a necessity measure but also a possibility to efficiently enumerate all extreme points. If we now extend our focus from necessity measures to arbitrary belief functions, then the analysis is more difficult, but Lemma 1 and Theorem 2 still hold. In the case of a necessity measure it was possible to look recursively at ascending focal sets and decide for every focal set if the corresponding mass should be assigned somewhere into the previous focal set (and then the previous focal set would already determine to which exact ω the mass should be assigned to actually obtain an extreme point) or if the mass should be assigned somewhere outside of the previous focal set and then every possible assignment would in fact lead to an extreme point.

If the focal sets are not nested then in the first place it is not clear with which focal set one should start some recursive procedure and how to proceed the recursion. But it is still possible to do a not too inefficient recursion that could generate a set of candidates of extreme points that actually includes all extreme points. One can (totally) order the focal sets in an arbitrary way that at least respects the order of set inclusion of the focal sets to make the recursion not unnecessarily ineffective. One possibility would be to order the focal sets according to their cardinality or another sort of rank function. (The linear ranking via cardinality is then not completely determined, so here comes some sort of arbitrariness into play). Then one could analogously go through ascending focal sets A_i and decide with the help of Theorem 2 to which state $\omega \in A_i$ the mass $m(A_i)$ should be assigned to actually obtain an extreme point. Then for a possible candidate of a selection λ that is already determined on the focal sets $A_1, \ldots, A_l$ one has to decide for the assignment of the mass $m(A_{l+1})$ to some $\omega^* \in A_{l+1}$ if this candidate ω^* is contained in some previous focal set $A \in \{A_1, \ldots, A_l\}$. If this is the case and if furthermore $\omega_\lambda(A) \in A_{l+1}$ and $\omega_\lambda(A) \neq \omega^*$ the assignment of the mass $m(A_{l+1})$ to this ω^* could be excluded, because it could not lead to an extreme point. (Note that in the case of a necessity measure it was enough to look only at the direct predecessor set A_l.)

We now shortly illustrate this recursive procedure via an example. Take $\Omega = \{\omega_1, \ldots, \omega_5\}$ and focal sets $A_1 = \{\omega_5\}$, $A_2 = \{\omega_4, \omega_5\}$, $A_3 = \{\omega_2, \omega_3, \omega_4\}$, $A_4 = \{\omega_2, \omega_3, \omega_4, \omega_5\}$. The indices indicate the ordering of the focal sets, here corresponding to the cardinality of the focal sets. In terms of focal sets this example is like the example above with the only exception that we added the focal set A_3 to make the focal sets not nested. As masses take for example $m(A_1) = 0.2$, $m(A_2) = 0.2$, $m(A_3) = 0.2$, $m(A_4) = 0.4$.

no.	$\omega_\lambda(A_i)$				P_λ				
1	5	5	4	5	0	0	0	0.2	0.8
2	5	5	3	5	0	0	0.2	0	0.8
3	5	5	3	3	0	0	0.6	0	0.4
4	5	5	2	5	0	0.2	0	0	0.8
5	5	5	2	2	0	0.6	0	0	0.4
6	5	4	4	4	0	0	0	0.8	0.2
7	5	4	3	3	0	0	0.6	0.2	0.2
8	5	4	2	2	0	0.6	0	0.2	0.2

Table 2: Summary of altogether 8 candidates of selections that could lead to extreme points.

Table 2 shows all 8 selections obtained by the recursive procedure that could possibly lead to extreme points. The second column describes the corresponding selections. For example the digit sequence 5535 means that the masses of A_1,A_2 and A_4 are assigned to ω_5 and the mass of A_3 is assigned to ω_3. This is similar to the digit sequence describing the characteristics in Figure 1, but note that for example selections 2 and 3 have the same characteristic and this is the only reason for choosing this description. The third column shows the 5 components of the corresponding extreme point candidates.

[12] We used the library lrslib, see http://cgm.cs.mcgill.ca/ avis/C/lrs.html.

Figure 4 shows the resulting core of the belief function for this example (black) together with the core of the necessity measure of the previous example (grey). One can see that compared to the necessity measure, the belief function has an extra facet and altogether 8 extreme points. In contrast to necessity measures, here for example the extreme points no. 1 and no. 8 differ only at two states but are not adjacent. Furthermore, for this example all 8 candidates of Table 2 are in fact extreme points, but this is generally not the case. A simple counterexample is $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and focal sets $A_1 = \{\omega_1, \omega_2\}$, $A_2 = \{\omega_1, \omega_3\}$, $A_3 = \{\omega_2, \omega_3\}$. Then Theorem 2 could not exclude any selection candidate. But for example with $m(A_1) = m(A_2) = m(A_3) = \frac{1}{3}$ and a selection with characteristic 132 we get an associated point $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. But this point is no extreme point of the core because it is a convex combination of the actual extreme points $p_1 = (0, \frac{2}{3}, \frac{1}{3})$ and $p_2 = (\frac{2}{3}, 0, \frac{1}{3})$ obtained by the selections with characteristics 232 and 113.

To exclude selections that do not lead to extreme points one can simultaneously consider the characterization of the extreme points given e.g. in [4, Proposition 9, p.274, Proposition 13, p.277]: Every extreme point of the core of a belief function (or even more generally a capacity of order 2) can be obtained via a total order $<$ on Ω and an associated selection λ that assigns all mass of a focal set A to the greatest element (w.r.t. $<$) of A. The selection with characteristic 132 of the above counterexample is obviously no λ associated to some total order $<$ because from $\omega_\lambda(\{\omega_1, \omega_2\}) = \omega_1$ it follows $\omega_2 < \omega_1$ and with $\omega_\lambda(\{\omega_1, \omega_3\}) = 3$ we have $\omega_1 < \omega_3$, so $\omega_2 < \omega_3$, but this is in contradiction with $\omega_\lambda(\{\omega_2, \omega_3\}) = \omega_2$. So with this "double description" of the extreme points one could exclude candidates of selections that do not lead to extreme points.[13] If we do this, then finally the question remains, if we possibly enumerate some of the extreme points more than one time with this modified procedure. Fortunately, we are able to show that this is not the case:

Theorem 6 *Let λ_1 and λ_2 be two different selections induced by some orderings $<_1$ and $<_2$ on Ω. Assume furthermore that for $i = 1, 2$ and for all focal sets A and A' the relation*

$$\{\omega_{\lambda_i}(A), \omega_{\lambda_i}(A')\} \subseteq A \cap A' \Longrightarrow \omega_{\lambda_i}(A) = \omega_{\lambda_i}(A')$$

of Theorem 2 is satisfied. Then the associated extreme points P_{λ_1} and P_{λ_2} are different.

Proof: Look at the (non-empty) system $D := \{A \in \mathscr{F}(\mathrm{Bel}) \mid \omega_{\lambda_1}(A) \neq \omega_{\lambda_2}(A)\}$. Then take that set $B \in D$ such that the associated $\omega_{\lambda_2}(B)$ is minimal w.r.t. $<_1$. Then the mass of B is transferred by λ_2 to $\omega := \omega_{\lambda_2}(B)$, so $P_{\lambda_2}(\{\omega\}) = \ldots + m(B) + \ldots$, but the mass of B is not transferred by λ_1 to ω. If $P_{\lambda_1}(\{\omega\}) = P_{\lambda_2}(\{\omega\})$ then there has to be another set $\tilde{B} \in D$ whose mass is transferred by λ_1 to ω but not by λ_2 to ω, so for the element $\tilde{\omega} := \omega_{\lambda_2}(\tilde{B}) \in \tilde{B}$ we have $\tilde{\omega} <_1 \omega$, but this is in contradiction to the minimality of $\omega_{\lambda_2}(B)$ w.r.t. $<_1$. So $P_{\lambda_1}(\{\omega\}) \neq P_{\lambda_2}(\{\omega\})$ and the two extreme points P_{λ_1} and P_{λ_2} are different. ■

[13] Another way to exclude transportations that do not lead to extreme points would be to check, if the selection is consistent in the sense of [5, p.25], cf. also Lemma 2 therein. Furthermore, also in the context of qualitative capacities the situation is similar, cf., [7, p.13].

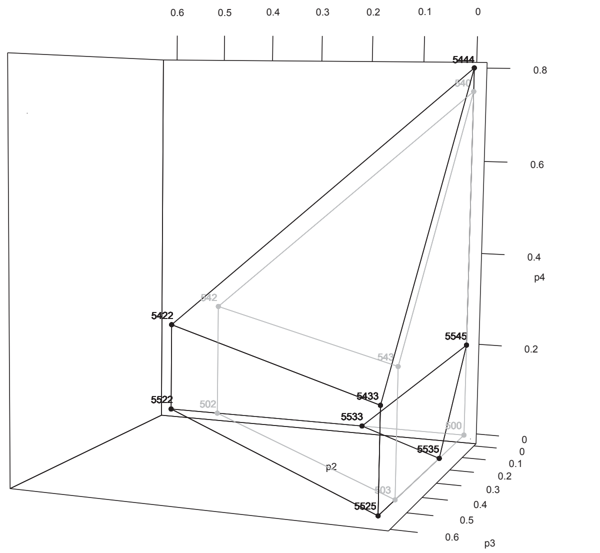

Figure 4: Comparison of the core of a necessity measure and a belief function.

With this we can efficiently enumerate the extreme points of an arbitrary belief function (on a finite space).

If the only task is to compute all extreme points, then another nice option of preprocessing could be simplifying in some situations: One could firstly factorize the space Ω according to the equivalence relation $\sim$ of indistinguishability: Two states ω and ω' are indistinguishable if every focal set A either contains both ω and ω' or contains neither ω nor ω'. Especially if there are only few focal sets on a big space Ω then the quotient space $W := \Omega_{/\sim}$ could be much smaller. One can then look at the associated belief function $\mathrm{Bel}_{/\sim} : 2^W \longrightarrow [0,1] : A \mapsto \mathrm{Bel}(\bigcup A)$ and compute in a first step the extreme points of $\mathrm{Bel}_{/\sim}$. The extreme points of the original belief function Bel can then be obtained by deciding in a second step for every extreme point $P_{/\sim}$ of $\mathrm{Bel}_{/\sim}$ and every equivalence class $w = [\omega]$ with $P_{/\sim}(\{w\}) > 0$ to which $\omega \in w$ the mass $P_{/\sim}(\{w\})$ assigned to the equivalence class w should be further assigned.

5 Conclusion

In this paper we worked out a combinatorial description of the extreme points of a necessity measure on a finite space. We treated necessity measures as special kinds of belief functions and were thus able to apply parts of our results also to arbitrary belief functions. Based on this we gave a possible procedure of seemingly efficiently enumerating the extreme points of belief functions.

For the case of arbitrary belief functions we did not explicitly analyze the complexity of enumeration procedures that use our results. This is a possible direction of further research.

Related to this there are a lot of further combinatorial questions. For instance: Is there a non-trivial bound for the number of extreme points in terms of the number of focal sets? Or: What is the maximal number of extreme points of a belief function where the set of focal elements builds an ordered set (w.r.t. set inclusion) that has a fixed width?[14]

Another direction of further research could be to analyze which parts of the given theorems and considerations of this paper still hold in the case of capacities of order 2 that are not belief functions.

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[14] The width of a partially ordered set $(X, \leq)$ is the maximal cardinality of a subset of elements that are all pairwise incomparable.