Weak Consistency for Imprecise Conditional Previsions

Renato Pelessoni
University of Trieste, Italy
renato.pelessoni@econ.units.it

Paolo Vicig
University of Trieste, Italy
paolo.vicig@econ.units.it

Abstract

In this paper we explore relaxations of (Williams) coherent and convex conditional previsions that form the families of n-coherent and n-convex conditional previsions, at the varying of n. We investigate which such previsions are the most general one may reasonably consider, suggesting (centered) 2-convex or, if positive homogeneity and conjugacy is needed, 2-coherent lower previsions. Basic properties of these previsions are studied. In particular, centered 2-convex previsions satisfy the Generalized Bayes Rule and always have a 2-convex natural extension. We discuss then the rationality requirements of 2-convexity and 2-coherence from a desirability perspective. Among the uncertainty concepts that can be modelled by 2-convexity, we mention generalizations of capacities and niveloids to a conditional framework.

Keywords. Williams coherence, 2-coherent previsions, 2-convex previsions, Generalized Bayes Rule.

1 Introduction

In his influential book [16], P. Walley developed a behavioural approach to imprecise probabilities (and previsions) extending de Finetti's [4] interpretation of precise previsions in terms of coherence. Operationally, this was achieved through a relaxation of de Finetti's betting scheme.

In fact, following de Finetti, P is a coherent precise prevision on a set $\mathcal{S}$ of gambles if and only if for all $m,\ n \in \mathbb{N}_0,\ s_1,\dots,s_m,r_1,\dots,r_n \geq 0,\ X_1,\dots,X_m,Y_1,\dots,Y_n \in \mathcal{S}$, defining $G = \sum_{i=1}^m s_i(X_i - P(X_i)) - \sum_{j=1}^n r_j(Y_j - P(Y_j))$, it holds that $\sup G \geq 0$. The terms $s_i(X_i - P(X_i))$, $r_j(Y_j - P(Y_j))$ are proportional (with coefficients or *stakes* s_i, r_j) to the *gains* arising from, respectively, buying X_i at $P(X_i)$ or selling Y_j at $P(Y_j)$. A coherent lower prevision $\underline{P}$ on $\mathcal{S}$ may be defined in a similar way, just restricting n to belong to $\{0,1\}$. This means that the betting scheme is modified to allow selling at most one gamble. Several other betting scheme variants have been investigated in the literature, either extending coherence for lower previsions (conditional lower previsions) or weakening it (previsions that are convex, or avoid sure loss). In particular, a convex lower prevision is defined introducing a convexity constraint $n = 1, \sum_{i=1}^m s_i = r_1 = 1$ in the betting scheme. In [16, Appendix B] n-coherent previsions are studied, as a different relaxation of coherence.

In this paper, we explore further variations of the behavioural approach/betting scheme: n-coherent and n-convex conditional lower previsions, formally defined later on as generalisations of the n-coherent (unconditional) previsions in [16]. Our major aims are:

a) to explore the flexibility of the behavioural approach and its capability to encompass different uncertainty models;

b) to point out which are the basic axioms/properties of coherence which hold even for much looser consistency concepts.

Referring to b) and with a view towards the utmost generality, we shall mainly concentrate on the extreme quantitative models that can be incorporated into a (modified) behavioural approach. This does not imply that these models should be regarded as preferable to coherent lower previsions. On the contrary they will not, as far as certain questions are concerned. For instance, inferences will typically be rather vague. However, it is interesting and somehow surprising to detect that certain properties like the Generalised Bayes Rule must hold even for such models, or that they can be approached in terms of desirability.

N-coherence and n-convexity may be naturally seen as relaxations of, respectively, (Williams) coherence and convexity. These and other preliminary concepts are recalled in Section 2. Starting from the weakest reasonably sound consistency concepts, we explore basic properties of 2-convex lower previsions in Section 3. We supply a characterisation by means of axioms, on a special set of conditional gambles generalising a linear space and termed $\mathcal{D}_{LIN}$ (Definition 2). Interestingly, it turns out that n-convexity with $n \geq 3$ and convexity are equivalent on $\mathcal{D}_{LIN}$. 2-convex previsions exhibit some drawbacks: a 2-convex natural extension may be defined, but its finiteness is not guaranteed; the property of internality may fail, as well as agreement with conditional implication (the Goodman-Nguyen relation). In Section 4, we show that the special subset of centered 2-convex previsions is not affected by these problems. In Section 5, 2-coherent lower previsions are discussed and characterised on $\mathcal{D}_{LIN}$ (Proposition 8). Again, n-coherence ($n \geq 3$) and coherence are equivalent on $\mathcal{D}_{LIN}$. On generic sets of gambles, n-coherent previsions ($n \geq 3$) have no n-coherent extension on sufficiently large supersets whenever the equivalence does not hold. We show also that 2-coherence should be preferred to 2-convexity when positive homogeneity and conjugacy are required. In Section 6 we analyse 2-convexity and 2-coherence in a desirability approach. Generalising prior work by Williams [17, 18] for coherence, we focus on the correspondence between these previsions and sets of desirable gambles, and on establishing the ensuing desirability rules. Models that can be accommodated into the framework of 2-convexity, but not of coherence, are presented in Section 7. These are conditional versions of capacities and niveloids. Section 8 concludes the paper. Due to spacing constraints, proofs of the results are omitted (some can be partly derived from results in [10, 12]).

2 Preliminaries

The starting points for our investigation are the known consistency concepts of coherent and convex lower conditional prevision [10, 11, 17, 18]. They both refer to an arbitrary set $\mathcal{D}$ of conditional gambles, that is of conditional bounded random variables. We denote with $X|B$ a generic conditional gamble, where X is a gamble and B is a non-impossible event ($B \neq \varnothing$). It is understood here that $X : I\!\!P \to \mathbb{R}$ is defined on an underlying partition $I\!\!P$ of atomic events ω, and that B belongs to the powerset of $I\!\!P$. Therefore, any $\omega \in I\!\!P$ implies either B or its negation $\neg B$ (in words, knowing that ω is true determines the truth value of B, i.e. B is known to be either true or false). Given B, the conditional partition $I\!\!P|B$ is formed by the conditional events $\omega|B$, such that ω implies B (implies that B is true) and $X|B : I\!\!P|B \to \mathbb{R}$ is such that $X|B(\omega|B) = X(\omega)$, $\forall \omega|B \in I\!\!P|B$. Because of this equality, several computations regarding $X|B$ can be performed by means of the restriction of X on B. In particular, it is useful for the sequel to recall that $\sup(X|B) = \sup_B X$, and $\inf(X|B) = \inf_B X$.

As special cases, we have that $X|\Omega = X$ is an unconditional gamble, $A|B$ a conditional event if A is an event (or its indicator I_A - we shall generally employ the same notation A for both).

As customary, a lower prevision $\underline{P}$ is, without further qualifications, a map from $\mathcal{D}$ into the real line, $\underline{P} : \mathcal{D} \to \mathbb{R}$. However, a lower prevision is often interpreted as a supremum buying price [16]. For instance, if a subject assigns $\underline{P}(X|B)$ to $X|B$, he is willing to buy X, conditional on B occurring, at any price lower than $\underline{P}(X|B)$. Under this behavioural interpretation, Definitions 1, 3, 5 check the consistency of $\underline{P}$, depending on whether it avoids losses bounded away from 0, according to different buying and selling constraints.

Definition 1. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be given.*

a) $\underline{P}$ is a coherent *conditional lower prevision on $\mathcal{D}$ iff, for all $m \in \mathbb{N}_0$, $\forall X_0|B_0, \dots, X_m|B_m \in \mathcal{D}$, $\forall s_0, \dots, s_m$ real and non-negative, defining $S(\underline{s}) = \bigvee\{B_i : s_i \neq 0, i = 0, \dots, m\}$ and $\underline{G} = \sum_{i=1}^m s_i B_i(X_i - \underline{P}(X_i|B_i)) - s_0 B_0(X_0 - \underline{P}(X_0|B_0))$, it holds, whenever $S(\underline{s}) \neq \varnothing$, that $\sup\{\underline{G}|S(\underline{s})\} \geq 0$.*

b) $\underline{P}$ is a convex *conditional lower prevision on $\mathcal{D}$ iff, for all $m \in \mathbb{N}^+$, $\forall X_0|B_0, \dots, X_m|B_m \in \mathcal{D}$, $\forall s_1, \dots, s_m$ real and non-negative such that $\sum_{i=1}^m s_i = 1$ (convexity constraint), defining $\underline{G}_c = \sum_{i=1}^m s_i B_i(X_i - \underline{P}(X_i|B_i)) - B_0(X_0 - \underline{P}(X_0|B_0))$, $S(\underline{s}) = \bigvee\{B_i : s_i \neq 0, i = 1, \dots, m\}$, it holds that $\sup\{\underline{G}_c|S(\underline{s}) \vee B_0\} \geq 0$.*

b1) $\underline{P}$ is centered convex or C-convex *on $\mathcal{D}$ iff it is convex and, $\forall X|B \in \mathcal{D}$, it is $0|B \in \mathcal{D}$ and $\underline{P}(0|B) = 0$.*

In the behavioural interpretation recalled above, Definition 1a) considers buying at most m conditional gambles $X_1|B_1, \dots, X_m|B_m$ (also no one, when $m = 0$) at prices $\underline{P}(X_1|B_1), \dots, \underline{P}(X_m|B_m)$, respectively, and selling at most one gamble $X_0|B_0$ at a supremum buying price $\underline{P}(X_0|B_0)$. The gain $\underline{G}$ is a linear combination with stakes $s_0, \dots, s_m$ of the gains from these transactions. It is conditioned on $S(\underline{s})$, to rule out both trivial transactions ($\underline{G} = 0$, since $s_1 = \ldots = s_m = 0$) and the case that $\underline{G} = 0$ because no transaction takes place (when $B_0, \dots, B_m$ are all false). Then, coherence requires the non-negativity of the supremum of $\underline{G}$, conditional on at least one non-trivial transaction being effective. The interpretation of Definition 1b) is similar: what changes is the convexity constraint on the stakes $(s_0 = 1), s_1, \dots, s_m$. This implies that $\underline{G}_c$ is the gain from one selling transaction and at least one buying transaction.

The definition of coherent lower prevision is a structure free version of Williams coherence, discussed in [11]. It is more general than Walley's coherence [16], in particular it always allows for a natural extension and is not necessarily conglomerable. The notion of convex lower prevision is still more general, and was introduced in [10], extending the unconditional convexity studied in [9]. Convex previsions can incorporate various uncertainty models, including convex risk measures, non-normalised possibility measures, and others. However, the special subclass of C-convex lower previsions guarantees better consistency properties. Among these, there always exists a convex natural extension of these measures, whose properties are analogous to those of the natural extension [10, Theorem 9].

Even though coherent and convex lower previsions can be defined on any set of conditional gambles, they are characterised by a few axioms on the special environment $\mathcal{D}_{LIN}$ defined next.

Definition 2. *Let $\mathcal{X}$ be a linear space of gambles and $\mathcal{B} \subset \mathcal{X}$ the set of all (indicators of) events in $\mathcal{X}$. Suppose $1 \in \mathcal{B}$ and $BX \in \mathcal{X}, \forall B \in \mathcal{B}, \forall X \in \mathcal{X}$. Setting $\mathcal{B}^{\varnothing} = \mathcal{B} - \{\varnothing\}$, define*

$$\mathcal{D}_{LIN} = \{X|B : X \in \mathcal{X}, B \in \mathcal{B}^{\varnothing}\}. \tag{1}$$

The sets $\mathcal{D}_{LIN}$ may be viewed as conditional generalisations of linear spaces of (unconditional) gambles. In fact, when $\mathcal{B} = \{\Omega, \varnothing\}$, $\mathcal{D}_{LIN}$ reduces to a linear space of unconditional gambles (including real constants). Not surprisingly then, characterisations on $\mathcal{D}_{LIN}$ have an unconditional counterpart on linear spaces.

Proposition 1. *Let $\underline{P} : \mathcal{D}_{LIN} \to \mathbb{R}$ be a conditional lower prevision.*

a) $\underline{P}$ is coherent on $\mathcal{D}_{LIN}$ if and only if [18]

(A1) $\underline{P}(X|B) - \underline{P}(Y|B) \le \sup\{X - Y|B\}$, $\forall X|B, Y|B \in \mathcal{D}_{LIN}$.[1]

(A2) $\underline{P}(\lambda X|B) = \lambda \underline{P}(X|B)$, $\forall X|B \in \mathcal{D}_{LIN}, \forall \lambda \ge 0$.

(A3) $\underline{P}(X + Y|B) \ge \underline{P}(X|B) + \underline{P}(Y|B)$, $\forall X|B,\ Y|B \in \mathcal{D}_{LIN}$.

(A4) $\underline{P}(A(X - \underline{P}(X|A \wedge B))|B) = 0$, $\forall X \in \mathcal{X}, \forall A, B \in \mathcal{B}^{\varnothing} : A \wedge B \neq \varnothing$.

b) $\underline{P}$ is convex on $\mathcal{D}_{LIN}$ if and only if (A1), (A4) and the following axiom hold [10, Theorem 8]

(A5) $\underline{P}(\lambda X + (1-\lambda)Y|B) \ge \lambda \underline{P}(X|B) + (1-\lambda)\underline{P}(Y|B), \forall X|B, Y|B \in \mathcal{D}_{LIN}, \forall \lambda \in]0,1[$.

[1] (A1) may be replaced by $\underline{P}(X|B) \ge \inf(X|B)$, $\forall X|B \in \mathcal{D}_{LIN}$, thus corresponding to the original version in [18].

Condition (A4) is the *Generalised Bayes Rule (GBR)*, introduced in [17, 18] and studied also in [16] in the special case $B = \Omega$.

Since our discussion will focus on minimal consistency properties for a conditional lower prevision, we have to mention a conditional generalisation of the implication (inclusion) relation between events, termed Goodman-Nguyen relation ($\le_{GN}$). In fact, suppose $A \Rightarrow B$ (or $A \subseteq B$). Then, asking that $\mu(A) \le \mu(B)$ is a really minimal rationality requirement for any μ aiming at measuring how likely an event is, given that, whenever event A will turn to be true, B will be true too. The following extension of the implication to conditional events was proposed in [8]:

$$A|B \le_{GN} C|D \quad \text{iff } A \wedge B \Rightarrow C \wedge D \\ \text{and } \neg C \wedge D \Rightarrow \neg A \wedge B. \tag{2}$$

The Goodman-Nguyen relation $\le_{GN}$ was extended to conditional gambles in [12]:

$$X|B \le_{GN} Y|D \text{ iff}$$
$$I_B X + I_{\neg B \vee D} \sup(X|B) \le I_D Y + I_{B \vee \neg D} \inf(Y|D)$$

showing that $X|B \le_{GN} Y|D$ implies $\underline{P}(X|B) \le \underline{P}(Y|D)$ for a C-convex or coherent $\underline{P}$ [12, Proposition 10].

3 2-Convex Lower Previsions

In Definition 1, a) and b), there is no upper bound to $m \in \mathbb{N}$. One may think of introducing it as a natural way of weakening coherence and convexity. More precisely, let us call *elementary gain* on $X_i|B_i$ any term $s_i B_i(X_i - \underline{P}(X_i|B_i))$, with the proviso that $-B_0(X_0 - \underline{P}(X_0|B_0))$ in Definition 1 b) is also an elementary gain, formally corresponding to $s_0 = -1$. Then, we may state that no more than n elementary gains are allowed in either $\underline{G}$ (Definition 1, a)) or $\underline{G}_c$ (Definition 1, b)). When doing so, we speak of *n-coherent* or *n-convex* lower previsions. This approach extends the notion of n-coherent (unconditional) prevision in [16, Appendix B].

Intuition suggests that the smaller n is, the more the corresponding consistency concept is looser. In the extreme cases n may be as small as 1 with coherence, 2 with convexity.

However, 1-*coherence* is too weak. In fact, $\underline{P}$ is 1-coherent on $\mathcal{D}$ iff, $\forall X_0|B_0 \in \mathcal{D}$, $\forall s_0 \in \mathbb{R}$, $\sup\{s_0 B_0(X_0 - \underline{P}(X_0|B_0))|B_0\} \ge 0$. It is easy to see that this is equivalent to *internality*, i.e. to requiring that $\underline{P}(X_0|B_0) \in [\inf(X_0|B_0), \sup(X_0|B_0)]$, $\forall X_0|B_0 \in \mathcal{D}$.

Since internality alone does not seem enough as a rationality requirement, we turn our attention in this section to what seems to be the next weakest consistency notion, that is 2-convexity.[2]

Definition 3. *$\underline{P} : \mathcal{D} \to \mathbb{R}$ is a* 2-convex *conditional lower prevision on $\mathcal{D}$ iff, $\forall X_0|B_0, X_1|B_1 \in \mathcal{D}$, we have that*

$$\sup\{B_1(X_1 - \underline{P}(X_1|B_1)) - B_0(X_0 - \underline{P}(X_0|B_0))|B_0 \vee B_1)\} \geq 0. \tag{3}$$

We explore now some basic features of 2-convex previsions. Some critical aspects are discussed next, showing in Section 4 that they can be solved resorting to the subclass of centered 2-convex previsions.

A remarkable result in our framework is the characterisation of 2-convexity on a structured set $\mathcal{D}_{LIN}$.

Proposition 2. *A conditional lower prevision $\underline{P} : \mathcal{D}_{LIN} \to \mathbb{R}$ is 2-convex on $\mathcal{D}_{LIN}$ if and only if (A1) and (A4) hold.*

To point out an important consequence of Proposition 2, compare it with Proposition 1 b). It follows at once that the difference between 2-convexity and convexity, on $\mathcal{D}_{LIN}$, is due to axiom (A5). On the other hand, the proof that a convex prevision on $\mathcal{D}_{LIN}$ must satisfy (A5), given in [10, Theorem 8], only involves a gain $\underline{G}_c$ made up of 3 elementary gains, i.e. it does not fully exploit convexity, but only 3-convexity. This justifies the following conclusion:

On $\mathcal{D}_{LIN}$, n-convexity with $n \geq 3$ and convexity are equivalent concepts.

Hence, the very difference between convexity and n-convexity reduces to that between convexity and 2-convexity, at least on $\mathcal{D}_{LIN}$. Yet, if $\underline{P}$ is defined on a set $\mathcal{D}$ other than $\mathcal{D}_{LIN}$, we may think of extending it to some $\mathcal{D}_{LIN} \supset \mathcal{D}$. If $\underline{P}$ is n-convex on $\mathcal{D}$, $n \geq 3$, and has an n-convex extension to $\mathcal{D}_{LIN}$, then $\underline{P}$ is convex on $\mathcal{D}_{LIN}$ and therefore also on $\mathcal{D}$. It ensues that if $\underline{P}$ is n-convex ($n \geq 3$) but not convex on $\mathcal{D}$, $\underline{P}$ will have no n-convex extension on any sufficiently large superset of $\mathcal{D}$ (any $\mathcal{D}^*$ including some $\mathcal{D}_{LIN}$ containing $\mathcal{D}$) - see also the later Example 2. This is a negative aspect of n-convexity, when $n \geq 3$. More generally, the discussion above shows that n-convex previsions are not particularly significant as an autonomous concept, when $n \geq 3$.

Turning again to 2-convex previsions, let us define a special extension, the 2-convex natural extension.

Definition 4. *Given a lower prevision $\underline{P} : \mathcal{D} \to \mathbb{R}$ and an arbitrary conditional gamble $Z|B$, let*

$$L(Z|B) = \{\alpha : \sup\{A(X - \underline{P}(X|A)) - B(Z - \alpha)|A \vee B\} < 0, \text{ for some } X|A \in \mathcal{D}\}. \tag{4}$$

Then the 2-convex natural extension *$\underline{E}_{2c}$ of $\underline{P}$ on $Z|B$ is*

$$\underline{E}_{2c}(Z|B) = \sup L(Z|B). \tag{5}$$

In general, $\underline{E}_{2c}(Z|B)$ may not be real-valued (i.e. $+\infty$, or $-\infty$ when $L(Z|B) = \emptyset$). The results in the next proposition are helpful in hedging this occurrence.

Proposition 3. *a) $L(Z|B) \neq \emptyset$, if there exists $Y|C \in \mathcal{D}$ such that $C \Rightarrow B$.*

b) Let $\underline{P}$ be 2-convex and such that $0|B \in \mathcal{D}$ and $\underline{P}(0|B) = 0$, $\forall X|B \in \mathcal{D}$. Given $0|C \notin \mathcal{D}$, the extension of $\underline{P}$ on $\mathcal{D} \cup \{0|C\}$ such that $\underline{P}(0|C) = 0$ is 2-convex.

c) When $L(Z|B) \neq \emptyset$, $L(Z|B) =]-\infty, \underline{E}_{2c}(Z|B)[$.

d) If $L(Z|B) \neq \emptyset$ and $\sup(X|A) \geq \underline{P}(X|A)$, $\forall X|A \in \mathcal{D}$, then $\underline{E}_{2c}(Z|B) \leq \sup(Z|B)$, $\forall Z|B$.

e) Let $\underline{P}$ be 2-convex and $0|B \in \mathcal{D}$, $\forall X|B \in \mathcal{D}$. Then, $\forall X|B \in \mathcal{D}$, $\sup(X|B) \geq \underline{P}(X|B)$ iff $\underline{P}(0|B) \leq 0$.

Parts a) and b) of Proposition 3 suggest a simple way to ensure $\underline{E}_{2c}(Z|B) \neq -\infty$: just add the gamble $0|B$ to $\mathcal{D}$, putting $\underline{P}(0|B) = 0$. To guarantee $\underline{E}_{2c}(Z|B) \neq +\infty$, it is sufficient that any $0|C$ in $\mathcal{D}$ (or added to $\mathcal{D}$) is given a non-positive lower prevision, by d) and e). Clearly, the simplest and most obvious choice is to put $\underline{P}(0|C) = 0$, $\forall 0|C$. This would make $\underline{P}$ a centered 2-convex lower prevision; in the remainder of this section we do not however rule out the possibility that $\underline{P}(0|C) \neq 0$ for some $0|C$.

The properties of the 2-convex natural extension are very similar to those of the natural extension:

Proposition 4. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a lower prevision, with $\mathcal{D} \subseteq \mathcal{D}_{LIN}$. If $\underline{E}_{2c}$ is finite on $\mathcal{D}_{LIN}$, then*

a) $\underline{E}_{2c}(X|B) \geq \underline{P}(X|B)$, $\forall X|B \in \mathcal{D}$.

b) $\underline{E}_{2c}$ is 2-convex on $\mathcal{D}_{LIN}$.

c) If $\underline{P}^*$ is 2-convex on $\mathcal{D}_{LIN}$ and $\underline{P}^*(X|B) \geq \underline{P}(X|B)$, $\forall X|B \in \mathcal{D}$, then $\underline{P}^*(X|B) \geq \underline{E}_{2c}(X|B)$, $\forall X|B \in \mathcal{D}_{LIN}$.

d) $\underline{P}$ is 2-convex on $\mathcal{D}$ if and only if $\underline{E}_{2c} = \underline{P}$ on $\mathcal{D}$.

e) If $\underline{P}$ is 2-convex on $\mathcal{D}$, $\underline{E}_{2c}$ is its smallest 2-convex extension on $\mathcal{D}_{LIN}$.

[2] 2-convex previsions were termed 1-convex in [1, 12]. Here we prefer the locution '2-convex' by analogy with the rule for fixing n in 'n-coherent' in [16].

In words, the 2-convex natural extension dominates $\underline{P}$ (by a)), characterises 2-convexity (by d)) and is the least-committal 2-convex extension of $\underline{P}$ (by b), c), e)).

Being rather weak a consistency concept, 2-convexity may not satisfy a number of properties which necessarily hold for coherent lower previsions. For instance, the *positive homogeneity* axiom (A2) of Proposition 1, $\underline{P}(\lambda X|B) = \lambda \underline{P}(X|B)$, with $\lambda \geq 0$, may not hold, not even weakening it to

$$\underline{P}(\lambda X|B) \geq \lambda \underline{P}(X|B), \forall \lambda \in [0,1]. \quad (6)$$

(Unconditional versions of (6) hold for centered convex previsions.)

It can instead be shown that

Proposition 5. *If, given $\lambda \in \mathbb{R}$, $\underline{P}$ is 2-convex on $\mathcal{D} \supseteq \{X|B, \lambda X|B\}$, then necessarily*

$$\inf\{(\lambda-1)X|B\} + \underline{P}(X|B) \leq \underline{P}(\lambda X|B) \leq \sup\{(\lambda-1)X|B\} + \underline{P}(X|B). \quad (7)$$

Condition (7) seems rather mild, as the next example points out.

Example 1. *Given $\mathcal{D} = \{X|B, 2X|B\}$ ($\lambda = 2$), where the image of $X|B$ is $[-1,1]$ and $\underline{P}(X|B) = 0.2$, equation (7) gives the bounds $\underline{P}(2X|B) \in [-0.8, 1.2]$. It is easy to check that $\underline{P}$ is 2-convex on $\mathcal{D}$ whatever is the choice for $\underline{P}(2X|B)$ in the interval $[-0.8, 1.2]$. According to the value for $\underline{P}(2X|B)$ selected in this interval, it may be $\underline{P}(2X|B) \gtreqless 2\underline{P}(X|B)$.*

An annoying feature of 2-convexity is that *internality* may fail, i.e. $\underline{P}(X|B)$ need not belong to the closed interval $[\inf(X|B), \sup(X|B)]$. Thus, 2-convex previsions may not satisfy a property holding even for 1-coherent previsions.

It has to be noticed that 2-convexity permits no complete freedom in departing from internality. There are two issues to be emphasized with respect to this question. The first tells us that lack of internality cannot be two-sided, because of the following result.

Proposition 6. *If $\underline{P} : \mathcal{D} \to \mathbb{R}$ is 2-convex on $\mathcal{D}$ and $\underline{P}(Y|D) < \inf(Y|D)$ for some $Y|D \in \mathcal{D}$, then $\underline{P}(X|B) \leq \sup(X|B)$, $\forall X|B \in \mathcal{D}$. Similarly, $\underline{P}(Y|D) > \sup(Y|D)$ for some $Y|D \in \mathcal{D}$ implies $\underline{P}(X|B) \geq \inf(X|B)$, $\forall X|B \in \mathcal{D}$.*

The second is the observation that 2-convexity imposes a sort of, so to say, two-component internality. To see this, note that

Lemma 1. *If $\underline{P} : \mathcal{D} \to \mathbb{R}$ is 2-convex on $\mathcal{D}$, and $X|B$, $Y|B \in \mathcal{D}$, then*

$$\inf\{X - Y|B\} \leq \underline{P}(X|B) - \underline{P}(Y|B) \leq \sup\{X - Y|B\}. \quad (8)$$

Recall now that $\underline{P}(X|B)$ is interpreted as a supremum buying price for $X|B$, and that Definition 3 ensures that buying $X|B$ for $\underline{P}(X|B)$ and selling $Y|B$ at its supremum buying price $\underline{P}(Y|B)$ would be (marginally) acceptable for 2-convexity. Then, equation (8) tells us that the profit $\underline{P}(X|B) - \underline{P}(Y|B)$ from this two-component exchange ($X|B$ vs. $Y|B$) guarantees no arbitrage. For instance, it cannot exceed $\sup\{X - Y|B\}$.

As a further critical issue with 2-convexity, we have that the Goodman-Nguyen relation may not induce an agreeing ordering on a 2-convex prevision. This is tantamount to saying that the partial ordering of some 2-convex conditional previsions may conflict with the ordering of the extended implication (inclusion) relation $\leq_{GN}$.

For instance, from (2), if $B \Rightarrow C$ then $0|C \leq_{GN} 0|B$. Agreement with the Goodman-Nguyen relation requires $\underline{P}(0|C) \leq \underline{P}(0|B)$ to hold, but it can be proven that if $\underline{P}(0|B) < 0$ and $B \Rightarrow C$, then 2-convexity asks instead that $\underline{P}(0|C) \geq \underline{P}(0|B)$ (the inequality may be strict).

4 Centered 2-Convex Lower Previsions

The critical issues on 2-convexity discussed in the preceding section can be solved or softened requiring the additional property

$$\forall X|B \in \mathcal{D}, 0|B \in \mathcal{D} \text{ and } \underline{P}(0|B) = 0, \quad (9)$$

i.e. restricting our attention to centered 2-convex conditional lower previsions. This is shown in the following proposition.

Proposition 7. *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a centered 2-convex lower prevision on $\mathcal{D}$. Then,*

a) $\forall X|B \in \mathcal{D}$, $\underline{P}(X|B) \in [\inf X|B, \sup X|B]$.

b) $\underline{P}$ has a finite 2-convex natural extension $\underline{E}_{2c}$ on any superset of $\mathcal{D}$.

c) $X|B \leq_{GN} Y|D$ implies $\underline{P}(X|B) \leq \underline{P}(Y|D)$.

Comment. The condition $\underline{P}(0|B) = 0$ appears as obvious, and in fact guarantees more satisfactory properties to 2-convexity. In our view, the main reason for considering the alternative $\underline{P}(0|B) \neq 0$ is to encompass additional uncertainty models. This is patent already in the unconditional framework: convex risk measures, as introduced in [6, 7], correspond to convex, not necessarily centered previsions [9].

Note that by Proposition 7 a) centered 2-convexity implies 1-coherence, while being obviously implied by

2-coherence. Hence, the centering condition $\underline{P}(0|B) = 0$ appears as a technical instrument to guarantee that the lower prevision $\underline{P}$ satisfies more properties than a generic 2-convex prevision, without having to assume the more demanding properties of 2-coherence.

5 2-Coherent Lower Previsions

Our next step is a discussion of which additional properties are achieved by 2-coherent lower prevision.

Definition 5. *$\underline{P} : \mathcal{D} \to \mathbb{R}$ is a 2-coherent lower prevision on $\mathcal{D}$ iff $\forall X_0|B_0, X_1|B_1 \in \mathcal{D}$, $\forall s_1 \geq 0$, $\forall s_0 \in \mathbb{R}$, defining $S(\underline{s}) = \bigvee\{B_i : s_i \neq 0, i = 0, 1\}$ we have that, whenever $S(\underline{s}) \neq \varnothing$,*

$$\sup\{s_1 B_1(X_1 - \underline{P}(X_1|B_1)) - s_0 B_0(X_0 - \underline{P}(X_0|B_0))|S(\underline{s})\} \geq 0. \tag{10}$$

2-coherent lower previsions are characterized on $\mathcal{D}_{LIN}$ as follows:

Proposition 8. *Let $\underline{P} : \mathcal{D}_{LIN} \to \mathbb{R}$ be a conditional lower prevision. $\underline{P}$ is 2-coherent on $\mathcal{D}_{LIN}$ if and only if (A1), (A2), (A4) and the following axiom hold:*

(A6) $\underline{P}(X|B) \leq -\underline{P}(-X|B)$.

Remark 1. *Proposition 8 can be equivalently restated replacing axiom (A1) with*

(A7) If $X|B, Y|B \in \mathcal{D}_{LIN}$, $\mu \in \mathbb{R}$ are such that $X|B \geq Y|B + \mu$, then $\underline{P}(X|B) \geq \underline{P}(Y|B) + \mu$.

In fact, it can be easily verified that (A1) and (A7) are equivalent.

Comment A comparison of Propositions 1 and 8 is useful in detecting at once two major differences between (centered) 2-convex and 2-coherent previsions.

One is positive homogeneity (axiom (A2)), a condition which, on any set $\mathcal{D}$, is necessary for 2-coherence, but not for 2-convexity. The need for positive homogeneity depends on the specific model we wish to consider. We might be willing to reject it in some instance, typically because of *liquidity risk* considerations. Basically, this means that for a large positive λ difficulties might be encountered at exchanging $\lambda X|B$ at a price $\underline{P}(\lambda X|B) = \lambda \underline{P}(X|B)$, because of lack of market liquidity at some degree.

The second difference is pointed out by axiom (A6). To fix its meaning, recall that given $\underline{P}(X|B)$, its *conjugate* upper prevision $\overline{P}(X|B)$ is defined by

$$\overline{P}(X|B) = -\underline{P}(-X|B). \tag{11}$$

Hence, by (11) axiom (A6) ensures that $\overline{P}(X|B) \geq \underline{P}(X|B)$, $\forall X|B \in \mathcal{D}_{LIN}$.

Therefore, 2-coherence is preferable to 2-convexity whenever we fix an upper ($\overline{P}$) and a lower ($\underline{P}$) bound for the uncertainty evaluation of $X|B$, while keeping positive homogeneity.

As an aside to the above discussion, we note that 2-coherence requires a weak form of homogeneity when $\lambda < 0$:

Proposition 9. *Given $\lambda < 0$, if $\underline{P}$ is 2-coherent on $D \supseteq \{\lambda X|B, X|B\}$, then necessarily $\underline{P}(\lambda X|B) \leq \lambda \underline{P}(X|B)$.*

Compare Propositions 8 and 1, a). Recalling that any 2-coherent lower prevision satisfies internality (being 1-coherent too), while (A6) is a necessary condition for coherence, only the superlinearity axiom (A3) distinguishes 2-coherence and coherence on $\mathcal{D}_{LIN}$. From this, deductions on the role of n-coherence, $n \geq 3$, can be made which are quite analogue to those on n-convexity in Section 3. This time, it can be shown that any n-coherent lower prevision, $n \geq 3$, must satisfy (A3), and hence that:

On $\mathcal{D}_{LIN}$, n-coherence with $n \geq 3$ and coherence are equivalent concepts.

And again, we may in general argue that n-coherence has no special relevance, compared to coherence, when $n \geq 3$. In particular, n-coherent extensions of an n-coherent $\underline{P}$ exist on sufficiently large sets if and only if $\underline{P}$ is coherent.

The latter concept is illustrated in the next example, elaborating on Example 2.7.6 in [16].

Example 2. *Let $I\!P = \{a, b, c, d\}$ be a partition of the sure event Ω. Define $\underline{P}$ on the powerset of $I\!P$ as follows:*

- $\underline{P}(\Omega) = 1$
- $\underline{P}(E) = \frac{1}{2}$ *if E is made up of 2 or 3 elements of $I\!P$, one of which is a.*
- $\underline{P}(E) = 0$ *otherwise.*

It is shown in [16] that $\underline{P}$ is not coherent, while being 3-coherent, and hence also 3-convex. We show now that $\underline{P}$ has no 3-convex extension to the linear space $\mathcal{L}(I\!P)$ of all gambles defined on $I\!P$.

In fact, suppose a 3-convex extension, also termed $\underline{P}$, exists, and define $A = a$, $B = a \vee b$, $C = a \vee c$, $D = a \vee d$. Note that, by applying (7) with $\lambda = \frac{1}{2}$, $X = A$ and $B = \Omega$, we get $\underline{P}(\frac{1}{2}A) \leq \underline{P}(A) = 0$. Therefore, also the 3-convex extension of $\underline{P}$ to $\frac{1}{4}(B + C + D - 1) = \frac{1}{2}A$ should be non-positive. However, by applying axiom

(A5) as a necessary condition of 3*-convexity and noting that* (7) *(with* $\lambda = -1$*,* $X = 1$ *and* $B = \Omega$*) ensures also that* $\underline{P}(-1) = -1$*, we obtain* $\underline{P}(\frac{1}{4}(B + C + D - 1)) = \underline{P}(\frac{1}{2}(\frac{1}{2}B + \frac{1}{2}C) + \frac{1}{2}(\frac{1}{2}D - \frac{1}{2})) \geq \frac{1}{2}\underline{P}(\frac{1}{2}B + \frac{1}{2}C) + \frac{1}{2}\underline{P}(\frac{1}{2}D - \frac{1}{2}) \geq \frac{1}{4}\underline{P}(B) + \frac{1}{4}\underline{P}(C) + \frac{1}{4}\underline{P}(D) + \frac{1}{4}\underline{P}(-1) \geq 3 \cdot \frac{1}{4} \cdot \frac{1}{2} - \frac{1}{4} = \frac{1}{8} > 0$*, a contradiction.*

From what we have just proven, we may conclude that:

a) the given $\underline{P}$ *on the powerset of* $\mathbb{P}$ *has no* 3*-convex extension to* $\mathcal{L}(\mathbb{P})$*;*

b) $\underline{P}$ *(viewed now as* 3*-coherent on the powerset of* $\mathbb{P}$*) has no* 3*-coherent extension on* $\mathcal{L}(\mathbb{P})$ *either: if it had one, this extension would be* 3*-convex too, contradicting a).*

We may thus conclude that centered 2-convexity and 2-coherence appear to be the most significant weakenings of (centered) convexity and coherence.

6 Weak Consistency in a Desirability Approach

In this section we examine centered 2-convexity and 2-coherence from the viewpoint of desirability. This is an alternative approach to rationality concepts for uncertainty measures going back to [17] in the case of conditional imprecise previsions. It has been recently applied to a variety of other situations, see e.g. the discussion in [13] and the results in [14].

Roughly speaking, a set $\mathcal{A}$ of gambles is considered.[3] It is such that its gambles are regarded as *desirable* or *acceptable*. We may in general be willing to establish some *rationality criteria*, requiring that certain gambles do, or do not, belong to $\mathcal{A}$. The basic problem we shall consider here is: which is the correspondence between the rationality criteria we adopt and the consistency concepts of centered 2-convexity or alternatively 2-coherence? More specifically, the following two questions arise:

Q1) Which rationality criteria should be required to the elements of a set $\mathcal{A}$, so that a conditional lower prevision $\underline{P}$ may be obtained from $\mathcal{A}$ that is 2-coherent (alternatively, 2-convex)?

Q2) Conversely, given a 2-coherent (alternatively, 2-convex) $\underline{P}$, does it determine a set $\mathcal{A}'$ with certain rationality properties?

In the case that $\underline{P}$ is coherent, the answer to Q1) and Q2) was given by Williams in [17]. Our approach to solving Q1) and Q2) was largely influenced by his work. Preliminarily, some notation must be introduced.

Definition 6. *Let* $\mathcal{X}$ *be a linear space of gambles,* $\mathcal{B} \subset \mathcal{X}$ *a set of (indicators of) events,* $\mathcal{B}^{\varnothing} = \mathcal{B} - \{\varnothing\}$*. We suppose* $\Omega \in \mathcal{B}$ *and* $BX \in \mathcal{X}$*,* $\forall B \in \mathcal{B}$*,* $\forall X \in \mathcal{X}$*.*[4] *Define then*

$$\begin{aligned} \mathcal{X}^{\succeq} &= \{X \in \mathcal{X} : \inf X \geq 0\}, \\ \mathcal{X}^{\preceq} &= \{X \in \mathcal{X} : \sup X \leq 0\}, \end{aligned} \tag{12}$$

and, $\forall B \in \mathcal{B}$*,*

$$\begin{aligned} \mathcal{R}(B) &= \{X \in \mathcal{X} : BX = X\}, \\ \mathcal{R}(B)^{\succ} &= \{X \in \mathcal{R}(B) : \inf\{X|B\} > 0\}, \\ \mathcal{R}(B)^{\prec} &= \{X \in \mathcal{R}(B) : \sup\{X|B\} < 0\}. \end{aligned} \tag{13}$$

If $\mathcal{S}$ *and* $\mathcal{T}$ *are subsets of* $\mathcal{X}$*, their* Minkowski sum *is*

$$\mathcal{S} + \mathcal{T} = \{X + Y : X \in \mathcal{S}, Y \in \mathcal{T}\}.$$

We shall use similar compact notation later. For instance, $\lambda\mathcal{S} + \mu\mathcal{T} \subseteq \mathcal{U}$*,* $\forall \lambda, \mu \geq 0$*, means:* $\forall X \in \mathcal{S}$*,* $\forall Y \in \mathcal{T}$*,* $\forall \lambda, \mu \geq 0$*,* $\lambda X + \mu Y \in \mathcal{U}$*.*

The following proposition answers question Q1) completely for 2-coherence:

Proposition 10. *Let* $\mathcal{A} \subseteq \mathcal{X}$ *be such that*

a) $\lambda\mathcal{A} + \mathcal{R}(B)^{\succ} \subseteq \mathcal{A}$*,* $\forall \lambda \geq 0$*,* $\forall B \in \mathcal{B}$*;*

b) $\mathcal{R}(B)^{\prec} \cap \mathcal{A} = \emptyset$*,* $\forall B \in \mathcal{B}$*.*

c) $(\mathcal{R}(B_1) \cap \mathcal{A}) + (\mathcal{R}(B_2) \cap \mathcal{A}) \subseteq \mathcal{R}(B_1 \vee B_2) \setminus \mathcal{R}(B_1 \vee B_2)^{\prec}, \forall B_1, B_2 \in \mathcal{B}$*.*

Define, $\forall X|B \in \mathcal{D}_{LIN}$*,*

$$\underline{P}(X|B) = \quad \sup\{x : B(X - x) \in \mathcal{A}\}. \tag{14}$$

Then, $\underline{P}$ *is* 2*-coherent on* $\mathcal{D}_{LIN}$*.*

Unlike the case of coherent conditional lower previsions examined in [17, Section 3.1], $\mathcal{A}$ does not need to be a cone in Proposition 10: given $X, Y \in \mathcal{A}$, $\lambda \geq 0$, neither $X + Y$ nor λX are guaranteed to belong to $\mathcal{A}$. Actually, condition a) represents a weakening of the cone axioms: if $X \in \mathcal{A}$, $Y \in \mathcal{R}(B)^{\succ}$ and $\lambda \geq 0$, then $\lambda X + Y \in \mathcal{A}$. This implies also $\mathcal{R}(B)^{\succ} \subseteq \mathcal{A}\ \forall B \in \mathcal{B}$, a condition that, like also b), is required for coherence as well (see (C1'), (C2') in [17, Section 3.1]).

The interpretation of b) is that of an *avoiding partial loss* condition: we can expect no gain from owning a gamble in $\mathcal{R}(B)^{\prec}$, when B is true, therefore such gambles cannot be included into $\mathcal{A}$.

[3] As will appear later, $\mathcal{A}$ is included into some fixed linear space of gambles.

[4] Note that if $X \in \mathcal{X}$ and $B \in \mathcal{B}^{\varnothing}$, $X|B \in \mathcal{D}_{LIN}$ in the notation of the preceding sections.

As for c), writing it in an extended form, it tells us that: if $X_1, X_2 \in \mathcal{A}$, $B_1X_1 = X_1$, $B_2X_2 = X_2$, then $(B_1 \vee B_2)(X_1+X_2) = X_1+X_2$ and $\sup(X_1+X_2|B_1\vee B_2) \geq 0$. Note that if $X_1 \in \mathcal{R}(B_1)$ and $X_2 \in \mathcal{R}(B_2)$, it always holds that $X_1 + X_2 \in \mathcal{R}(B_1 \vee B_2)$, without having to impose it by means of axiom c). In fact, we have that $(B_1 \vee B_2)(X_1 + X_2) = (B_1 \vee B_2)(B_1X_1 + B_2X_2) = (B_1 \vee B_2)B_1X_1 + (B_1 \vee B_2)B_2X_2 = B_1X_1 + B_2X_2 = X_1 + X_2$, so that $X_1 + X_2 \in \mathcal{R}(B_1 \vee B_2)$.

Therefore, the essential condition in axiom c) is that if X_1, X_2 are desirable (belonging to $\mathcal{A}$), this does not imply that $X_1 + X_2 \in \mathcal{A}$ (which is required for coherence in [17, 18]), but only that $X_1 + X_2$ is not necessarily discarded by resorting to b). To illustrate this concept, let for instance $B_1 = B_2 = \Omega$ in c), so that $\mathcal{R}(B_1) = \mathcal{R}(B_2) = \mathcal{R}(B_1 \vee B_2) = \mathcal{R}(\Omega) = \mathcal{X}$. Then, c) implies $X_1+X_2 \notin \mathcal{R}(\Omega)^{\prec}$, making impossible to apply b) in order to discard $X_1 + X_2$ from $\mathcal{A}$.

As for question Q2), an answer is given by the following proposition, when $\underline{P}$ is 2-coherent.

Proposition 11. *Let $\underline{P} : \mathcal{D}_{LIN} \to \mathbb{R}$ be 2-coherent. Define*

$$\mathcal{A}' = \{\lambda B(X - x) + Y : X|B \in \mathcal{D}_{LIN}, \\ x < \underline{P}(X|B), Y \in \mathcal{X}^{\succeq}, \lambda \geq 0\}. \tag{15}$$

Then the set $\mathcal{A}'$ is such that:

a') $a\mathcal{A}' + \mathcal{X}^{\succeq} \subseteq \mathcal{A}'$, $\forall a \geq 0$;

b') $\mathcal{X}^{\preceq} \cap \mathcal{A}' = \{0\}$;

c') $(\mathcal{A}' + \mathcal{A}') \setminus \{0\} \subseteq \mathcal{X} \setminus \mathcal{X}^{\preceq}$;

d') $\underline{P}(X|B) = \sup\{x : B(X - x) \in \mathcal{A}'\}$, $\forall X|B \in \mathcal{D}_{LIN}$.

Proposition 11 states the existence of a set of desirable gambles $\mathcal{A}'$, in accordance with a given 2-coherent conditional lower prevision $\underline{P}$ and satisfying the rationality criteria a'), b'), c'). Comparing a'), b') with a), b) in Proposition 10, a clear similarity comes evident: essentially, the sets $\mathcal{R}(B)^{\succ}$, $\mathcal{R}(B)^{\prec}$, $B \in \mathcal{B}$, have been replaced with $\mathcal{X}^{\succeq}$, $\mathcal{X}^{\preceq}$ respectively. As a consequence, note that $0 \in \mathcal{A}'$.

The interpretation of c') is similar to c) in Proposition 10. It tells that: if $X_1, X_2 \in \mathcal{A}'$, $X_1 + X_2 \neq 0$, then $\sup(X_1 + X_2) > 0$. Again, coherence would allow the stronger implication $X_1, X_2 \in \mathcal{A}' \to X_1+X_2 \in \mathcal{A}'$, while 2-coherence only ensures that $X_1 + X_2$ does not belong to the (near) rejection set $\mathcal{X}^{\preceq}$.

Actually, a'), b') c') prove to be stronger than a), b), c). This means that any 2-coherent conditional prevision can be represented through a set of desirable gambles $\mathcal{A}'$ satisfying the necessary axioms a'), b'), c'), but also that, at the same time, $\mathcal{A}'$ satisfies the weaker axioms a), b), c) in Proposition 10.

A comparison between (3) in Definition 3 and (10) in Definition 5 intuitively suggests that we can get an answer to Q1) for 2-convexity from a reduced form of Proposition 10, with $\lambda = 1$. More precisely, the following proposition holds:

Proposition 12. *Let $\mathcal{A} \subseteq \mathcal{X}$ be such that*

a) $\mathcal{A} + \mathcal{R}(B)^{\succ} \subseteq \mathcal{A}$, $\forall B \in \mathcal{B}$;

b) $\mathcal{R}(B)^{\prec} \cap \mathcal{A} = \emptyset$, $\forall B \in \mathcal{B}$.

Define, $\forall X|B \in \mathcal{D}_{LIN}$,

$$\underline{P}(X|B) = \sup\{x : B(X - x) \in \mathcal{A}\}. \tag{16}$$

Then, $\underline{P}$ is 2-convex on $\mathcal{D}_{LIN}$. Moreover, $\underline{P}$ is centered iff $\mathcal{R}(B)^{\succ} \subseteq \mathcal{A}$ $\forall B \in \mathcal{B}$.

An analogously reduced form of Proposition 11 allows us to answer question Q2) for 2-convexity.

Proposition 13. *Let $\underline{P} : \mathcal{D}_{LIN} \to \mathbb{R}$ be 2-convex. Define*

$$\mathcal{A}' = \{B(X - x) + Y : X|B \in \mathcal{D}_{LIN}, \\ x < \underline{P}(X|B), Y \in \mathcal{X}^{\succeq}\}. \tag{17}$$

Then the set $\mathcal{A}'$ is such that:

a) $\mathcal{A}' + \mathcal{X}^{\succeq} \subseteq \mathcal{A}'$;

b) $\mathcal{X}^{\preceq} \cap \mathcal{A}' = \emptyset$ iff $\underline{P}(0|B) \leq 0$, $\forall B \in \mathcal{B}^{\varnothing}$;

c) $\underline{P}(X|B) = \sup\{x : B(X - x) \in \mathcal{A}'\}$, $\forall X|B \in \mathcal{D}_{LIN}$.

Further, $\underline{P}$ is centered iff $\mathcal{R}(B)^{\succ} \subseteq \mathcal{A}'$ $\forall B \in \mathcal{B}$.

Comparing Propositions 10 and 11 with, respectively, Propositions 12 and 13, we note that, in addition to the constraint $\lambda = 1$, 2-convexity requires no condition like c) and c') in Propositions 10 and 11 respectively. Referring, for instance, to c'), this means that, given $X, Y \in \mathcal{A}'$ with $X + Y \neq 0$, 2-convexity does not guarantee $\sup(X + Y) > 0$: summing up two individually desirable gambles could therefore give rise to a partial or even to a sure loss. Moreover, a non-centered 2-convex $\underline{P}$ suffers from a more serious shortcoming: if $\mathcal{R}(B)^{\succ} \subseteq \mathcal{A}'$ does not necessarily hold, then a non-negative gamble $X = BX$ $(X \neq 0)$ exists, that is considered non-desirable. The main drawbacks of 2-convexity relative to 2-coherence are therefore clearly pointed out by a comparison through desirability axioms.

7 Weakly Consistent Uncertainty Models

As mentioned in the Introduction, a motivation for studying the loose forms of consistency introduced in this paper is their capability of encompassing or extending uncertainty models already investigated in the literature. Even though these models may depart also considerably from coherence and convexity, they can nevertheless be accommodated into a unifying betting scheme, ranging from 2-convex to coherent lower previsions.

Focusing on 2-convexity, we first recall a few definitions and some results concerning unconditional 2-convex lower previsions.

Definition 7. *Given a finite partition $\mathbb{P}$, and denoting with $2^{\mathbb{P}}$ its powerset, a mapping $c: 2^{\mathbb{P}} \to [0,1]$ is a (normalised)* capacity *whenever $c(\varnothing) = 0$, $c(\Omega) = 1$ (normalisation) and $\forall A_1, A_2 \in 2^{\mathbb{P}}$ such that $A_1 \Rightarrow A_2$, $c(A_1) \leq c(A_2)$ (1-monotonicity).*

Definition 8. *Given a linear space $\mathcal{L}$ of random variables, a* niveloid *[2, 5] is a functional $N : \mathcal{L} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ which is translation invariant and monotone, i.e. such that*

$$\begin{aligned} &N(X+\mu) = N(X) + \mu, \forall X \in \mathcal{L}, \forall \mu \in \mathbb{R}; \\ &X \geq Y \text{ implies } N(X) \geq N(Y), \forall X, Y \in \mathcal{L}. \end{aligned} \tag{18}$$

As well-known, capacities are uncertainty measures with really minimal quantitative requirements. Niveloids can be viewed as a generalisation of theirs to linear spaces of random variables which preserves their minimality properties. Strictly speaking, this is true for centered niveloids, i.e. such that $N(0) = 0$. In fact, the centering condition $N(0) = 0$ does not ensue from the definition of niveloid. Note also that niveloids apply to random variables which may be unbounded too.

It has been proven in [1, Section 4.1][5] that:

Proposition 14. *a) Let $\underline{P}$ be defined on $2^{\mathbb{P}}$. Then $\underline{P}$ is a centered 2-convex lower prevision if and only if it is a capacity.*

b) Let $\underline{P}$ be defined on a linear space $\mathcal{L}$ of bounded random variables (gambles). Then $\underline{P}$ is a 2-convex lower prevision if and only if it is a (finite-valued) niveloid.

Hence, an unconditional 2-convex lower prevision is equivalent to a capacity or a niveloid, on structured sets ($2^{\mathbb{P}}$ or $\mathcal{L}$ respectively). On non-structured sets,ù it extends these concepts.

[5] See Footnote 2.

2-convex conditional lower previsions are natural candidates to define conditional capacities and niveloids on *arbitrary* sets of, respectively, conditional events or gambles. To the best of our knowledge, such conditional versions have not been considered yet in this general conditional environment, but rather in more specific cases. For instance, [3] focuses on updating rules for 'convex' capacities, which means for 2-monotone lower probabilities, while considering a single conditioning event.

Thus 2-convex previsions may provide an appropriate framework for such extensions, guaranteeing some minimal properties like the existence of a 2-convex natural extension (when being centered). Take for instance centered 2-convex conditional lower probabilities. They satisfy the properties one would require to a conditional capacity: $\underline{P}(0|B) = 0$, $\underline{P}(\Omega|B) = 1$ (this follows from Proposition 7, a)), and $A|B \leq_{GN} C|D$ implies $\underline{P}(A|B) \leq \underline{P}(C|D)$ (Proposition 7, c)). Similarly, centered 2-convex lower previsions ensure generalisations of properties (18) (see especially Proposition 2 and Remark 1 for the first property, Proposition 7, c) for the second).

8 Conclusions

N-convex and n-coherent conditional lower previsions broaden the spectrum of uncertainty measures that can be accommodated into a behavioural approach to imprecision, including, for instance, conditional extensions of capacities and niveloids when $n = 2$. This choice for n is the most neatly distinguished from coherence, the other extreme in the spectrum, and that retaining more interesting properties. Among these the GBR must still hold. Centered 2-convex and 2-coherent previsions also have a clear meaning in terms of desirability. Further work is necessary to investigate additional properties, like the possible existence of envelope theorems, or properties of already defined notions. In particular, we conjecture that the 2-convex natural extension may simplify computing the convex natural extension. As a further generalisation of this work, the consistency notions defined here could be extended to the case of unbounded conditional random variables. This has been done in [15] for coherent conditional lower previsions, while, to the best of our knowledge, a similar investigation for convex conditional previsions is still missing.

9 Acknowledgments

We wish to thank the referees for their helpful suggestions. We acknowledge partial support by the FRA2013 grant 'Models for Risk Evaluation, Uncer-

tainty Measurement and Non-Life Insurance Applications'.

References

[1] P. Baroni, R. Pelessoni and P. Vicig. Generalizing Dutch risk measures through imprecise previsions. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 17(2):153–177, 2009.

[2] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci and A. Rustichini. Niveloids and their extensions: Risk measures on small domains. *Journal of Mathematical Analysis and Applications*, 413(1):343–360, 2014.

[3] A. Chateauneuf, R. Kast and A. Lapied. Conditioning capacities and Choquet integrals: the role of comonotony. *Theory and Decision*, 51(2–4):367–386, 2001.

[4] B. de Finetti. *Theory of Probability*. Wiley, 1974.

[5] S. Dolecki and G.H. Greco. Niveloids. *Topological Methods in Nonlinear Analysis*, 5:1–22, 1995.

[6] H. Föllmer and A. Schied. Convex measures of risk and trading constraints. *Finance and Stochastics*, 6:429–447, 2002.

[7] H. Föllmer and A. Schied. Robust preferences and convex measures of risk. In: *Advances in Finance and Stochastics*, edited by K. Sandmann, K. and P. J. Schönbucher, 39–56, Springer-Verlag, 2002.

[8] I.R. Goodman and H.T. Nguyen. Conditional objects and the modeling of uncertainties. In: *Fuzzy Computing*, edited by M. Gupta and T. Yamakawa, 119–138, Elsevier (North-Holland) 1988.

[9] R. Pelessoni and P. Vicig. Convex imprecise previsions. *Reliable Computing*, 9:465–485, 2003.

[10] R. Pelessoni and P. Vicig. Uncertainty modelling and conditioning with convex imprecise previsions. *International Journal of Approximate Reasoning*, 39(2–3):297–319, 2005.

[11] R. Pelessoni and P. Vicig. Williams coherence and beyond. *International Journal of Approximate Reasoning*, 50(4):612–626, 2009.

[12] R. Pelessoni and P. Vicig. The Goodman–Nguyen relation within imprecise probability theory. *International Journal of Approximate Reasoning*, 55(8):1694–1707, 2014.

[13] E. Quaeghebeur. Desirability. In: *Introduction to Imprecise Probabilities*, edited by T. Augustin, F.P.A. Coolen, G. de Cooman, M.C.M. Troffaes, 1–27, J. Wiley and Sons, 2014.

[14] E. Quaeghebeur, G. de Cooman and F. Hermans. Accept & reject statement-based uncertainty models. *International Journal of Approximate Reasoning*, 57:69–102, 2015.

[15] M. C .M. Troffaes and G. de Cooman. *Lower Previsions*. Wiley, 2014.

[16] P. Walley. *Statistical reasoning with imprecise probabilities*. Chapman and Hall, 1991.

[17] P. M. Williams. *Notes on conditional previsions*. Research Report, School of Math. and Phys. Science, University of Sussex, 1975.

[18] P. M. Williams. Notes on conditional previsions. *International Journal of Approximate Reasoning*, 44(3):366–383,2007.