Credal Compositional Models and Credal Networks

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Abstract

This paper studies the composition operator for credal sets introduced at the last ISIPTA conference in more detail. Our main attention is devoted to the relationship between a special type of compositional model, so-called perfect sequences of credal sets, and those of (precise) probability distributions, with the goal of finding the relationship between credal compositional models and credal networks. We prove that a perfect sequence of credal sets is a convex hull of perfect sequences of extreme points of these credal sets. Finally, we reveal the relationship among credal networks (in a general sense), perfect sequences of credal sets and separately specified credal networks.

Keywords. Credal sets, strong independence, credal networks, separate specification, compositional models.

1 Introduction

The most widely used models managing uncertainty and multidimensionality are, at present, the so-called *probabilistic graphical Markov models*. The problem of multidimensionality is solved in these models with the help of the concept of conditional independence, which enables factorisation of a multidimensional probability distribution into small parts (marginals, conditionals or just factors). Among them, the most popular are Bayesian networks. Therefore, it is not very surprising that analogous models have also been studied in several theories of imprecise probability [1, 2, 3].

Credal networks represent a generalisation of Bayesian networks capable of dealing with imprecision. Compositional models for credal sets, on the other hand, are intended to be a generalisation of compositional models for precise probabilities [6, 7, 8]. As the equivalence between Bayesian networks and precise compositional models is well known [9], it also seems quite natural to ask a similar question in this more general case.

Compositional models have also been introduced in possibility theory [13, 14] (where these models are parameterised by a continuous t-norm) and a few years ago in evidence theory [10, 11] as well. In all these frameworks the original idea is preserved but certain slight differences between them are present.

Although Bayesian networks and (precise) probabilistic compositional models represent the same class of distributions, they do not do it in the same way. Namely, Bayesian networks use *conditional distributions*, whereas compositional models consist of *unconditional distributions*. Naturally, both types of models contain the same information but, while some marginal distributions are explicitly expressed in compositional models, it may happen that their computation from the corresponding Bayesian network is rather computationally expensive.

Furthermore, the research concerning the relationship between compositional models in evidence theory and evidential networks [15] revealed an aspect that is probably even more important. Even though any evidential network (with a proper conditioning rule and conditional independence concept) can be expressed as a compositional model, if we do it in the opposite way and transform a compositional model into an evidential network, we may realise that the model is more imprecise than the original one. This is caused by the fact that conditioning increases imprecision.

In [16] we introduced a composition operator for credal sets, but due to the problem of discontinuity it needs a revision. This task seems to be rather difficult and has not been satisfactorily solved yet. Therefore, we decided to postpone its definition for the general case to the future and now we deal only with the case of projective credal sets, as this approach is sufficient for the topic of this paper.

The goal of this paper is to show that the composition operator for credal sets is worth developing, as compositional models seem to be a reasonable counterpart of credal networks. We prove that the perfect sequence of credal sets is a convex hull of perfect sequences of extreme points of these credal sets. We prove that any separately specified credal network can be expressed in the form of a perfect sequence of credal sets, and any perfect sequence of credal sets can be expressed as a credal network (in a general sense). Finally, we present an algorithm for transforming a compositional model to a credal network.

This contribution is organized as follows. In Section 2 we summarise the basic concepts and notation. Definition of the operator of composition is recalled in Section 3, which is completely devoted to its basic properties and those of compositional models. Finally, in Section 4 the relationship between credal networks and compositional models is studied.

2 Basic Concepts and Notation

In this section we will recall the basic concepts and notation necessary for understanding the paper.

2.1 Variables and Distributions

For an index set $N = \{1, 2, \ldots, n\}$ let $\{X_i\}_{i \in N}$ be a system of variables, each X_i having its values in a finite set $\mathbf{X}_i$, and $\mathbf{X}_N = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n$ be the Cartesian product of these sets.

In this paper we will deal with groups of variables on subspaces of the Cartesian product. Let us note that X_K will denote a group of variables $\{X_i\}_{i \in K}$ with values in

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i$$

throughout the paper.

Any group of variables X_K can be described by a *probability distribution* (sometimes also called *probability function*)

$$P : \mathbf{X}_K \longrightarrow [0, 1],$$

such that $\sum_{x_K \in \mathbf{X}_K} P(x_K) = 1$.

Having two probability distributions P_1 and P_2 of X_K, we say that P_1 is *absolutely continuous* with respect to P_2 (and denote $P_1 \ll P_2$) if for any $x_K \in \mathbf{X}_K$

$$P_2(x_K) = 0 \Longrightarrow P_1(x_K) = 0.$$

This concept plays an important role in the definition of the composition operator.

2.2 Credal Sets

A *credal set* $\mathcal{M}(X_K)$ describing a group of variables X_K is defined as a closed convex set of probability measures describing the values of these variables.[1]

[1] For $K = \emptyset$ let us set $\mathcal{M}(X_\emptyset) \equiv 1$.

In order to simplify the expression of operations with credal sets, it is often considered [12] that a credal set is the set of probability distributions associated with the probability measures in it. Under such consideration, a credal set can be expressed as a *convex hull* of its extreme distributions

$$\mathcal{M}(X_K) = \mathrm{CH}\{\mathrm{ext}(\mathcal{M}(X_K))\}.$$

Consider a credal set describing X_K, i.e., $\mathcal{M}(X_K)$. For each $L \subset K$ its *marginal credal set* $\mathcal{M}(X_L)$ is obtained by element-wise marginalisation, i.e.,

$$\mathcal{M}(X_L) = \mathrm{CH}\{P^{\downarrow L} : P \in \mathrm{ext}(\mathcal{M}(X_K))\}, \quad (1)$$

where $P^{\downarrow L}$ denotes the marginal distribution of P on $\mathbf{X}_L$.

Having two credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ describing X_K and X_L, respectively (assuming that $K, L \subseteq N$), we say that these credal sets are *projective* if their marginals describing the common variables coincide, i.e., if

$$\mathcal{M}_1(X_{K \cap L}) = \mathcal{M}_2(X_{K \cap L}). \quad (2)$$

Let us note that if K and L are disjoint, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are always projective, as $\mathcal{M}_1(X_\emptyset) = \mathcal{M}_2(X_\emptyset) \equiv 1$.

Conditional credal sets are obtained from the joint ones by point-wise conditioning of the extreme points and subsequent linear combination of the resulting conditional distributions. More formally: Let $\mathcal{M}(X_{K \cup L})$ ($K \cap L = \emptyset$) be a credal set describing (groups of) variables $X_{K \cup L}$. Then for any $x_L \in \mathbf{X}_L$

$$\mathcal{M}(X_K | x_L) = \mathrm{CH}\{P(X_K | x_L) : P \in \mathrm{ext}(\mathcal{M}(X_{K \cup L}))\}, \quad (3)$$

is a *conditional credal set* describing X_K given $X_L = x_L$.

2.3 Strong Independence

Among numerous definitions of independence for credal sets [4] we have chosen strong independence, as it seems to be the most appropriate for multidimensional models.

We say that (groups of) variables X_K and X_L (K and L disjoint) are *strongly independent* with respect to $\mathcal{M}(X_{K \cup L})$ iff (in terms of probability distributions)

$$\mathcal{M}(X_{K \cup L}) = \mathrm{CH}\{P_1 \cdot P_2 : P_1 \in \mathrm{ext}(\mathcal{M}(X_K)), P_2 \in \mathrm{ext}(\mathcal{M}(X_L))\}. \quad (4)$$

Again, several generalisations of this notion to conditional independence already exist, see, e.g., [12], but since the following definition is suggested by the authors as the most appropriate for the marginal problem, it seems to be a suitable concept in our case as well, since the composition operator can also be used as a tool for solving the marginal problem, as shown (within the framework of possibility theory), e.g., in [14].

Given three groups of variables X_K, X_L and X_M (where K, L, M are mutually disjoint subsets of N such that K and L are nonempty), we say in a way analogous[2] to [12] that X_K and X_L are *conditionally strongly independent* given X_M under the global set $\mathcal{M}(X_{K\cup L\cup M})$ (we will denote this relationship by $K \perp\!\!\!\perp L|M$) iff

$$\begin{aligned}\mathcal{M}(X_{K\cup L\cup M}) \\ = \mathrm{CH}\{(P_1 \cdot P_2)/P_1^{\downarrow M} : P_1 \in \mathrm{ext}(\mathcal{M}(X_{K\cup M})), \\ P_2 \in \mathrm{ext}(\mathcal{M}(X_{L\cup M})), P_1^{\downarrow M} = P_2^{\downarrow M}\}. \quad (5)\end{aligned}$$

This definition is a generalisation of stochastic conditional independence: if $\mathcal{M}(X_{K\cup L\cup M})$ is a singleton, then $\mathcal{M}(X_{K\cup M})$ and $\mathcal{M}(X_{L\cup M})$ are also (projective) singletons and the definition is reduced to the definition of stochastic conditional independence.

3 Compositional Models

In this section we will summarise the achieved results concerning compositional models for credal sets. Most of them are presented without proofs; missing proofs can be found in [16]. The concept of the composition operator is presented first in a precise probability framework, as it seems to be useful for better understanding to the concept.

3.1 Composition Operator and Its Properties

Now, let us recall the definition of composition of two credal sets. Consider two index sets $K, L \subset N$. We do not put any restrictions on K and L; they may be but need not be disjoint, and one may be a subset of the other.

In order to enable the reader to understand this concept, let us first present the definition of composition for precise probabilities [6]. Let P_1 and P_2 be two probability distributions of (groups of) variables X_K and X_L; then

$$(P_1 \triangleright P_2)(X_{K\cup L}) = \frac{P_1(X_K) \cdot P_2(X_L)}{P_2(X_{K\cap L})}, \qquad (6)$$

whenever $P_1(X_{K\cap L}) \ll P_2((X_{K\cap L})$; otherwise, it remains undefined.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be credal sets describing X_K and X_L, respectively. Our original goal in [16] was to define a new credal set, denoted by $\mathcal{M}_1 \triangleright \mathcal{M}_2$, which will be describing $X_{K\cup L}$ and will contain all of the information contained in $\mathcal{M}_1$ and, as much as possible, in $\mathcal{M}_2$.

The required properties are met by Definition 1 in [16][3]. However, the definition exhibits a kind of discontinuity and should be reconsidered. Therefore, we will only deal with the composition of projective credal sets in this paper.

Definition 1 For two projective credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ describing X_K and X_L, their *composition* $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is defined by the following expression:

$$\begin{aligned}(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L}) \\ = \mathrm{CH}\{(P_1 \cdot P_2)/P_2^{\downarrow K\cap L} : P_1 \in \mathrm{ext}(\mathcal{M}_1(X_K)), \\ P_2 \in \mathrm{ext}(\mathcal{M}_2(X_L)), P_1^{\downarrow K\cap L} = P_2^{\downarrow K\cap L}\}.\end{aligned}$$

The following lemma, proven in [16], contains basic properties possessed by this composition operator.

Lemma 1 *For two projective credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ describing X_K and X_L, respectively, the following properties hold true:*

(i) $\mathcal{M}_1 \triangleright \mathcal{M}_2$ *is a credal set describing* $X_{K\cup L}$.

(ii) $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_K) = \mathcal{M}_1(X_K)$ *and* $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_L) = \mathcal{M}_2(X_L)$.

(iii) $\mathcal{M}_1 \triangleright \mathcal{M}_2 = \mathcal{M}_2 \triangleright \mathcal{M}_1$.

As the operator is, at present, defined only for projective sets, it is commutative, as suggested by *(iii)* of this lemma. Furthermore, it follows from *(ii)* that the operator keeps both marginals. Both of these properties are typical in other settings exactly for the case of projective marginals.

Despite these facts, it remains non-associative (in general), as can be seen from the following example.

Example 1 Let X_1 and X_2 be two binary variables and

$$\mathcal{M}_1(X_1) = \mathrm{CH}\{[0.2, 0, 8], [0.5, 0.5]\}$$

and

$$\mathcal{M}_2(X_2) = \mathrm{CH}\{[0.3, 0.7], [0.6, 0.4]\}$$

[2] Let us note that our definitions somehow differ from those presented in [12]: the authors there require point-wise satisfaction in (4) and (5), which leads to non-convexity. In [5], this type of independence is called *complete*.

[3] Let us note that the definition is based on Moral's concept of conditional independence with relaxing convexity.

be two credal sets describing X_1 and X_2, respectively; further let

$$\mathcal{M}_3(X_1X_2) = \text{CH}\{[0.2, 0, 0.1, 0.7], [0.5, 0, 0.1, 0.4]\}$$

be another credal set describing both X_1 and X_2. Here $[a, b]$ means $P(x_1) = a$ and $P(\bar{x}_1) = b$, and similarly $[a, b, c, d]$ means $P(x_1x_2) = a$, $P(x_1\bar{x}_2) = b$, $P(\bar{x}_1x_2) = c$ and $P(\bar{x}_1\bar{x}_2) = d$.

Using (1) to $\mathcal{M}_3(X_1X_2)$, one can realise that both $\mathcal{M}_1(X_1)$ and $\mathcal{M}_2(X_2)$ are marginal to $\mathcal{M}_3(X_1X_2)$.

$\mathcal{M}_1 \triangleright \mathcal{M}_2$ is obtained via Definition 1:

$$\begin{aligned}(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2) \\ = \text{CH}\{&[0.06, 0.14, 0.24, 0.56], [0.12, 0.48, 0.08, 0.32] \\ &[0.15, 0.35, 0.15, 0.35], [0.3, 0.2, 0.3, 0.2]\},\end{aligned}$$

but $\mathcal{M}_1 \triangleright \mathcal{M}_2$ cannot be composed with $\mathcal{M}_3$, as they are not projective. On the other hand

$$\begin{aligned}(\mathcal{M}_2 \triangleright \mathcal{M}_3)(X_1X_2) \\ = \text{CH}\{[0.2, 0, 0.1, 0.7], [0.5, 0, 0.1, 0.4]\},\end{aligned}$$

as follows from (ii) of Lemma 1 and similarly, for the same reason,

$$\begin{aligned}(\mathcal{M}_1 \triangleright (\mathcal{M}_2 \triangleright \mathcal{M}_3))(X_1X_2) \\ = \text{CH}\{[0.2, 0, 0.1, 0.7], [0.5, 0, 0.1, 0.4]\}. \quad \diamondsuit\end{aligned}$$

The following theorem, also proven in [16], expresses the relationship between strong independence and the operator of composition. It is, together with Lemma 1, the most important assertion enabling us to introduce multidimensional models.

Theorem 1 *Let $\mathcal{M}$ be a credal set describing $X_{K\cup L}$ with marginals $\mathcal{M}(X_K)$ and $\mathcal{M}(X_L)$. Then*

$$\mathcal{M}(X_{K\cup L}) = (\mathcal{M}^{\downarrow K} \triangleright \mathcal{M}^{\downarrow L})(X_{K\cup L})$$

iff

$$(K \setminus L) \perp\!\!\!\perp (L \setminus K) | (K \cap L).$$

3.2 Perfect Sequences of Credal Sets

In this subsection we will recall repetitive application of the composition operator with the goal to create a multidimensional model. Since the operator is not associative, as demonstrated in Example 1, we have to specify in which order the low-dimensional credal sets are composed together. To make the formulae more transparent, we will omit parentheses in the case the operator is to be applied from left to right, i.e., in what follows

$$\begin{aligned}&\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3 \triangleright \cdots \triangleright \mathcal{M}_{m-1} \triangleright \mathcal{M}_m \\ &\quad = (\cdots((\mathcal{M}_1 \triangleright \mathcal{M}_2) \triangleright \mathcal{M}_3) \triangleright \cdots \triangleright \mathcal{M}_{m-1}) \triangleright \mathcal{M}_m.\end{aligned} \tag{7}$$

Furthermore, we will always assume $\mathcal{M}_i$ to be a credal set describing X_{K_i} and call $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots, \mathcal{M}_m$ a *generating sequence* of model (7).

The reader familiar with some papers on probabilistic, possibilistic or evidential compositional models knows that one of the most important notions in this theory is that of a so-called *perfect sequence*, already introduced in [16] also for credal sets. Let us recall it here.

Definition 2 A generating sequence of credal sets $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ is called *perfect* if

$$\begin{aligned}\mathcal{M}_1 \triangleright \mathcal{M}_2 &= \mathcal{M}_2 \triangleright \mathcal{M}_1, \\ \mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3 &= \mathcal{M}_3 \triangleright (\mathcal{M}_1 \triangleright \mathcal{M}_2), \\ &\vdots \\ \mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_m &= \mathcal{M}_m \triangleright (\mathcal{M}_1 \triangleright \cdots \triangleright \mathcal{M}_{m-1}).\end{aligned}$$

Let us note that the concept of perfect sequence of probability distributions is a special case of this definition, in the case of all credal sets being singletons.

It is evident that the necessary condition for perfectness is pairwise projectivity (i.e., (2) holds for any pair of credal sets from the generating sequence in question) of low-dimensional credal sets. However, from Example 1 one can easily see that this condition need not be sufficient.

Therefore a stronger, necessary and sufficient condition, expressed by the following assertion, must be fulfilled.

Lemma 2 *A generating sequence $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$ is perfect* iff *the pairs of credal sets $\mathcal{M}_j$ and $(\mathcal{M}_1 \triangleright \cdots \triangleright \mathcal{M}_{j-1})$ are projective, i.e., if*

$$\begin{aligned}&\mathcal{M}_j(X_{K_j\cap(K_1\cup\cdots\cup K_{j-1})}) \\ &\quad = (\mathcal{M}_1 \triangleright \cdots \triangleright \mathcal{M}_{j-1})(X_{K_j\cap(K_1\cup\cdots\cup K_{j-1})}),\end{aligned}$$

for all $j = 2, 3, \ldots, m$.

From Definition 2 one can hardly identify the properties of perfect sequences beyond the algebraic ones; the most important one is expressed by the following characterisation theorem, which also suggests why these sequences are called perfect.

Theorem 2 *A generating sequence of credal sets $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$ is perfect* iff *all the credal sets from this sequence are marginal to the composed credal set $\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_m$:*

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_m)(X_{K_j}) = \mathcal{M}_j(X_{K_j}),$$

for all $j = 1, \ldots, m$.

The following (almost trivial) assertion, which brings the sufficient condition for perfectness, resembles assertions concerning decomposable models.

Theorem 3 *Let a generating sequence of pairwise projective credal sets $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$ be such that $K_1, K_2, \ldots, K_m$ satisfies the following running intersection property:*

$$\forall j = 2, 3, \ldots, m \;\; \exists \ell (1 \leq \ell < j) \quad \text{such that} \quad K_j \cap (K_1 \cup \cdots \cup K_{j-1}) \subseteq K_\ell.$$

Then the sequence $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$ is perfect.

It should be emphasised that the running intersection property of $K_1, K_2, \ldots, K_m$ is a sufficient condition to guarantee perfectness of a generating sequence of pairwise projective assignments. By no means is this condition necessary, as already demonstrated in [16].

Therefore, not only is perfectness of a sequence a structural property connected with the properties of $K_1, K_2, \ldots, K_m$ but it also depends on specific values of the respective basic assignments.

3.3 Perfect Sequence as Convex Hull

In this subsection we will study the relationship between perfect sequences of credal sets and those of a probability distribution. Before doing that, let us present a simple lemma necessary for the proof of the main theorem.

Lemma 3 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two projective credal sets describing X_K and X_L, respectively. Then*

$$\begin{aligned}\{\text{ext}((\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_K \cup X_L))\} &\subseteq \{P_1 \triangleright P_2 : P_1 \in \text{ext}(\mathcal{M}_1(X_K)), \\ &\quad P_2 \in \text{ext}(\mathcal{M}_2(X_L)), P_1^{\downarrow K\cap L} = P_2^{\downarrow K \cap L}\}.\end{aligned} \tag{8}$$

Proof. By Definition 1, $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L})$ is the convex hull of the set of probability distributions from the set on the right-hand side of (8), taking into account the definition of the composition operator for precise probabilities. Therefore its extreme points must also belong to this set. □

Equality need not hold in (8), as can be seen from the following simple example.

Example 2 Let

$$\mathcal{M}_1(X_1) = \text{CH}\{[0.2, 0.8], [0.5, 0.5]\}$$

and

$$\mathcal{M}_2(X_2) = \text{CH}\{[0.5, 0.5], [0.8, 0.2]\}$$

be two credal sets describing X_1 and X_2, respectively. Then, as mentioned above, $\mathcal{M}_1(X_1)$ and $\mathcal{M}_2(X_2)$ are projective, and therefore $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is obtained by Definition 1:

$$\begin{aligned}(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_1X_2) &= \text{CH}\{[0.1, 0.4, 0.1, 0.4], [0.16, 0.04, 0.64, 0.16], \\ &\quad [0.25, 0.25, 0.25, 0.25], [0.4, 0.1, 0.4, 0.1]\},\end{aligned} \tag{9}$$

nevertheless $[0.25, 0.25, 0.25, 0.25]$ is not an extreme point of (9) because it can be obtained as a linear combination of $[0.1, 0.4, 0.1, 0.4]$ and $[0.4, 0.1, 0.4, 0.1]$. ♢

Theorem 4 *Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$ be a perfect sequence of credal sets such that each $\mathcal{M}_i, i = 1, \ldots m$, is the convex hull of its extreme points, i.e.,*

$$\mathcal{M}_i(X_{K_i}) = \text{CH}\{P_i : P_i \in \text{ext}(\mathcal{M}_i(X_{K_i}))\}.$$

Then

$$\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_m$$

is a convex hull of all

$$P_1 \triangleright P_2 \triangleright \ldots \triangleright P_m$$

such that each $P_i \in \text{ext}(\mathcal{M}_i(X_{K_i}))$, and $P_1, P_2, \ldots, P_m$ form a perfect sequence.

Proof. Let us prove the assertion by induction. For $m = 2$ it is obvious as it follows directly from Definition 1. Let us suppose that

$$\begin{aligned}\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_j &= \text{CH}\{P_1 \triangleright P_2 \triangleright \ldots \triangleright P_j, P_i \in \text{ext}(\mathcal{M}_i), \\ &\quad P_1, P_2, \ldots, P_j \text{ is perfect}\}\end{aligned}$$

for $2 \leq j < m$ and prove that

$$\begin{aligned}\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_{j+1} &= \text{CH}\{P_1 \triangleright P_2 \triangleright \ldots \triangleright P_{j+1}, P_i \in \text{ext}(\mathcal{M}_i), \\ &\quad P_1, P_2, \ldots, P_{j+1} \text{ is perfect}\}\end{aligned} \tag{10}$$

holds as well.

By convention (7)

$$\begin{aligned}\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \ldots \triangleright \mathcal{M}_j \triangleright \mathcal{M}_{j+1} &= (\cdots \mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_j) \triangleright \mathcal{M}_{j+1}\end{aligned}$$

and since $\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \ldots \triangleright \mathcal{M}_j$ and $\mathcal{M}_{j+1}$ are projective, we can apply Definition 1 to these credal sets to obtain

$$\begin{aligned}(\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_j) \triangleright \mathcal{M}_{j+1} &= \text{CH}\{Q_j \cdot \frac{P_{j+1}}{P_{j+1}^{\downarrow (K_1 \cup \cdots \cup K_j) \cap K_{j+1}}}, \\ &\quad Q_j \in \text{ext}(\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_j), \\ &\quad P_{j+1} \in \text{ext}(\mathcal{M}_{j+1}), \\ &\quad Q_j^{\downarrow (K_1 \cup \cdots \cup K_j) \cap K_{j+1}} = P_{j+1}^{\downarrow (K_1 \cup \cdots \cup K_j) \cap K_{j+1}}\}.\end{aligned}$$

However, due to Lemma 3

$$Q_j \in \{P_1 \triangleright P_2 \triangleright \ldots \triangleright P_j, P_i \in \mathrm{ext}(\mathcal{M}_i), \\ P_1, P_2, \ldots, P_j \text{ is perfect}\}.$$

Let us denote by $P_1^*, P_2^*, \ldots, P_j^*$ a perfect sequence such that

$$Q_j = P_1^* \triangleright P_2^* \triangleright \ldots \triangleright P_j^*.$$

Then, due to Lemma 2 (applied to precise probability distributions) $P_1^*, P_2^*, \ldots, P_j^*, P_{j+1}$ forms a perfect sequence. Therefore

$$\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_{j+1} \\ \subseteq \mathrm{CH}\{P_1 \triangleright P_2 \triangleright \ldots \triangleright P_{j+1}, P_i \in \mathrm{ext}(\mathcal{M}_i), \\ P_1, P_2, \ldots, P_{j+1} \text{ is perfect}\}.$$

Let, on the other hand, $P_1, P_2, \ldots, P_{j+1}$ be a perfect sequence of distributions such that each $P_i \in \mathrm{ext}(\mathcal{M}_i)$. Then

$$P_1 \triangleright P_2 \triangleright \ldots \triangleright P_{j+1} \in \mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_{j+1},$$

and therefore also

$$\mathrm{CH}\{P_1 \triangleright P_2 \triangleright \ldots \triangleright P_{j+1}, P_1, P_2, \ldots, P_{j+1} \text{ is perfect}\} \\ \subseteq \mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_{j+1}.$$

Therefore (10) is satisfied. □

Example 3 Let $\mathcal{M}_1(X_1)$ and $\mathcal{M}_2(X_2)$ be the two credal sets from Example 2,

$$\mathcal{M}_3(X_1X_2X_3) \\ = \mathrm{CH}\{[0.1, 0, 0.3, 0.1, 0.05, 0.05, 0.1, 0.3], \\ [0.16, 0, 0.03, 0.01, 0.32, 0.32, 0.04, 0.12], \\ [0.4, 0, 0.075, 0.025, 0.2, 0.2, 0.025, 0.075]\}$$

and

$$\mathcal{M}_4(X_3X_4) \\ = \mathrm{CH}\{[0.44, 0.11, 0.18, 0.27], [0.56, 0.14, 0.12, 0.18], \\ [0.33, 0.22, 0.09, 0.36], [0.42, 0.28, 0.06, 0.24]\}.$$

These credal sets form a perfect sequence $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$, since $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is marginal to $\mathcal{M}_3$, and $\mathcal{M}_3$ and $\mathcal{M}_4$ are projective, as from (1) one gets

$$\mathcal{M}_3(X_3) = \mathrm{CH}\{[0.55, 0.45], [0.7, 0.3]\} = \mathcal{M}_4(\mathrm{X}_3).$$

The credal set $\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3 \triangleright \mathcal{M}_4(X_1, X_2, X_3, X_4)$ is then expressed as

$$\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3 \triangleright \mathcal{M}_4 \tag{11} \\ = \mathrm{CH}\{[0.08, 0.02, 0, 0, 0.24, 0.06, 0.04, 0.06, 0.04, \\ 0.01, 0.02, 0.03, 0.08, 0.02, 0.12, 0.18], \\ [0.06, 0.04, 0, 0, 0.18, 0.12, 0.02, 0.08, 0.03, \\ 0.02, 0.01, 0.04, 0.06, 0.04, 0.06, 0.24], \\ [0.128, 0.032, 0, 0, 0.024, 0.006, 0.004, \\ 0.006, 0.256, 0.064, 0.128, 0.192, \\ 0.032, 0.008, 0.048, 0.072], \\ [0.096, 0.064, 0, 0, 0.018, 0.012, 0.002, \\ 0.008, 0.192, 0.128, 0.064, 0.256, \\ 0.024, 0.016, 0.024, 0.096], \\ [0.32, 0.08, 0, 0, 0.06, 0.015, 0.015, 0.01, 0.16, \\ 0.04, 0.08, 0.12, 0.02, 0.005, 0.03, 0.015], \\ [0.24, 0.16, 0, 0, 0.045, 0.03, 0.005, 0.02, 0.12, \\ 0.08, 0.04, 0.16, 0.015, 0.01, 0.015, 0.06]\}.$$

This credal set can be obtained either directly by successive application of Definition 1 or as a convex hull of $P_1^{i_1} \triangleright P_2^{i_2} \triangleright P_3^{i_3} \triangleright P_4^{i_4}$, where any $P_1^{i_1}, P_2^{i_2}, P_3^{i_3}, P_4^{i_4}$ forms a perfect sequence, and any $P_j^{i_j} \in \mathrm{ext}(\mathcal{M}_j)$. In this example we have six perfect sequences, namely

$$\begin{array}{ll} P_1^1, P_2^1, P_3^1, P_4^1; & P_1^1, P_2^1, P_3^1, P_4^3; \\ P_1^1, P_2^2, P_3^2, P_4^1; & P_1^1, P_2^2, P_3^2, P_4^3; \\ P_1^2, P_2^2, P_3^3, P_4^2; & P_1^2, P_2^2, P_3^3, P_4^4, \end{array} \tag{12}$$

where

$$P_1^1 = [0.2, 0.8], \qquad P_1^2 = [0.5, 0.5], \\ P_2^1 = [0.5, 0.5], \qquad P_2^2 = [0.8, 0.2], \\ P_3^1 = [0.1, 0, 0.3, 0.1, 0.05, 0.05, 0.1, 0.3], \\ P_3^2 = [0.16, 0, 0.03, 0.01, 0.32, 0.32, 0.04, 0.12], \\ P_3^3 = [0.4, 0, 0.075, 0.025, 0.2, 0.2, 0.025, 0.075], \\ P_4^1 = [0.44, 0.11, 0.18, 0.27], \\ P_4^2 = [0.56, 0.14, 0.12, 0.18], \\ P_4^3 = [0.33, 0.22, 0.09, 0.36], \\ P_4^4 = [0.42, 0.28, 0.06, 0.24]. \qquad \diamondsuit$$

As we stated in the Introduction, in the precise probability framework any multidimensional distribution representable by a Bayesian network can also be represented in the form of a perfect sequence, and vice versa. An analogous result, although somewhat weaker, for perfect sequences of credal sets will be presented in the next section.

4 Credal Networks

In this section we will deal with credal networks and their relationship to credal compositional models.

4.1 Basic Concepts

A *credal network* [1] over X_N is (in analogy to Bayesian networks) a pair $(\mathcal{G}, \{\mathbf{P}^1, \ldots, \mathbf{P}^k\})$ such that, for any

$i = 1, \dots, k$, $(\mathcal{G}, \mathbf{P}^i)$, is a Bayesian network over X_N, i.e., each $\mathbf{P}^i$ is a system of conditional probability distribution forming the joint distribution of X_N, $P^i(X_N)$.

The resulting model is a credal set, which is the convex hull of the Bayesian networks, i.e.,

$$\mathrm{CH}\{P^1(X_N), \dots, P^k(X_N)\}.$$

It is evident that this definition loses the attractiveness of Bayesian networks, where the overall information is computed from local pieces of information. Let us denote by $\mathcal{CN}(X_N)$ the class of all credal networks over X_N.

The most popular (and also most effective) type of credal networks is represented by those called separately specified. A *separately specified credal network* over X_N is a pair $(\mathcal{G}, \mathbf{M})$, where $\mathbf{M}$ is a set of conditional credal sets $\mathcal{M}(X_i|pa(X_i))$ for each $X_i \in X_N$, and $pa(X_i)$ denotes the *set of parent variables* of X_i. Here the overall model is, in analogy to Bayesian networks, obtained as a strong extension of the $\mathcal{M}(X_i|pa(X_i)), i \in N$. Analogous to the previous paragraph, let us denote by $\mathcal{SCN}(X_N)$ the class of all separately specified credal networks over X_N.

Nevertheless, a lot of situations exist in which separately specified credal networks either cannot be used or their use leads to less specific models. For more details, the reader is referred to [1]; one extremely simple example can be found in the next subsection (Example 5).

4.2 Credal Networks and Perfect Sequences of Credal Sets

In this subsection we will prove, using the preceding results, a relationship between credal networks and perfect sequences of credal sets. For this purpose, let us denote by $\mathcal{CM}(X_N)$ the class of compositional models over X_N.

Theorem 5 *For any X_N*

$$\mathcal{SCN}(X_N) \subset \mathcal{CM}(X_N) \subset \mathcal{CN}(X_N). \tag{13}$$

Proof. Let

$$(\mathcal{G}, \mathcal{M}(X_i|pa(X_i)), i \in N) \tag{14}$$

be a separately specified credal network over X_N and N be ordered in such a way that $i > j \in pa(i)$ for each $i \in N$. The overall model (joint credal set describing X_N) is then obtained as a strong extension of credal sets from (14).

Let us define $\mathcal{M}_i(X_i \cup pa(X_i))$ as a strong extension of $\mathcal{M}(X_i|pa(X_i))$ and $\mathcal{M}(pa(X_i))$, where $\mathcal{M}(pa(X_i))$ is a marginal of the strong extension of $\mathcal{M}(X_j|pa(X_j)), j = 1, \dots, i-1$. Now it easily follows that any $\mathcal{M}_i(X_i \cup pa(X_i))$ is a marginal of the strong extension of (14). Therefore, credal sets $\mathcal{M}_1(X_1), \dots, \mathcal{M}_n(X_i \cup pa(X_n))$ form a perfect sequence defining the same joint model as (14).

If $\mathcal{M}_1(X_{K_1}), \dots, \mathcal{M}_m(X_{K_m})$ is perfect, then according to Theorem 4

$$\begin{aligned}\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_m &= \mathrm{CH}\{P_1 \triangleright P_2 \triangleright \ldots \triangleright P_m, P_i \in \mathrm{ext}(\mathcal{M}_i),\\ &\qquad P_1, P_2, \dots, P_m \text{ is perfect}\}.\end{aligned}$$

For any perfect sequence $P_1, P_2, \dots, P_m$ a Bayesian network exists representing the distribution

$$P_1 \triangleright \ldots \triangleright P_m$$

such that, for each variable X_j, $\ell \in \{1, \dots, m\}$ exists such that $(\{X_j\} \cup pa(X_j)) \subset \{X_i\}_{i \in K_\ell}$. Therefore,

$$\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_m = \mathrm{CH}\{(\mathcal{G}^{\mathrm{i}}, \mathbf{P}^{\mathrm{i}}), 1, \dots, \mathrm{k}\}.$$

As any perfect sequence represents the same system of conditional independences, it is evident that any Bayesian network can be defined on the same graph $\mathcal{G}$, which concludes the proof. □

For the description of an algorithm reconstructing a credal network from a perfect sequence of credal sets the reader is referred to the following subsection.

The following simple examples demonstrate that the inclusions in (13) are proper.

Example 4 Let X_1 and X_2 be two binary variables and P_1 and P_2 be defined as follows

$$\begin{array}{lll} P_1(x_1) = 0.4 & P_1(x_2|x_1) = 0.25 & P_1(x_2|\bar{x}_1) = 0.5,\\ P_2(x_1) = 0.6 & P_2(x_2|x_1) = 0.5 & P_2(x_2|\bar{x}_1) = 0.25.\end{array}$$

They form, together with the graph $X_1 \longrightarrow X_2$, two Bayesian networks. The corresponding credal network is

$$\mathrm{CH}\{[0.1, 0.3, 0.3, 0.3], [0.3, 0.3, 0.1, 0.3]\}. \tag{15}$$

From these distributions one can get the following credal sets forming a perfect sequence

$$\begin{aligned}\mathcal{M}_1(X_1) &= \mathrm{CH}\{[0.4, 0.6], [0.6, 0.4]\},\\ \mathcal{M}_2(X_1X_2) &= \mathrm{CH}\{[0.1, 0.3, 0.3, 0.3], [0.2, 0.2, 0.15, 0.45]\}\\ &\qquad \{[0.15, 0.45, 0.2, 0.2], [0.3, 0.3, 0.1, 0.3]\}.\end{aligned}$$

It is evident that $\mathcal{M}_1 \triangleright \mathcal{M}_2(X_1X_2) = \mathcal{M}_2(X_1X_2)$, which also contains other Bayesian networks not contained in (15). ♢

Example 5 Let

$$\mathcal{M}_1(X_1X_2) = \text{CH}\{[0.2, 0.2, 0, 0.6], [0.1, 0.4, 0.1, 0.4], [0.25, 0.25, 0.25, 0.25], [0.2, 0.3, 0.3, 0.2]\}.$$

be a credal set describing variables X_1 and X_2 with values in $\mathbf{X}_1$ and $\mathbf{X}_2$ ($\mathbf{X}_i = \{x_i, \bar{x}_i\}$), respectively.

From its extreme points we obtain the following distributions:

$$\begin{array}{lll} P_1(x_2) = 0.2 & P_1(x_1|x_2) = 1 & P_1(x_1|\bar{x}_2) = 0.25 \\ P_2(x_2) = 0.2 & P_2(x_1|x_2) = 0.5 & P_2(x_1|\bar{x}_2) = 0.5 \\ P_3(x_2) = 0.5 & P_3(x_1|x_2) = 0.5 & P_3(x_1|\bar{x}_2) = 0.5 \\ P_4(x_2) = 0.5 & P_4(x_1|x_2) = 0.4 & P_2(x_1|\bar{x}_2) = 0.6. \end{array}$$

These are, together with the graph $X_2 \longrightarrow X_1$, four Bayesian networks. Their convex hull is exactly the set $\mathcal{M}_1(X_1X_2)$. Nevertheless, it is not a separately specified credal network. To obtain that, we need the credal sets $\mathcal{M}(X_2)$, $\mathcal{M}(X_1|x_2)$ and $\mathcal{M}(X_1|\bar{x}_2)$.

Using (1) and (3), we obtain

$$\begin{aligned} \mathcal{M}(X_2) &= \text{CH}\{[0.2, 0.8], [0.5, 0.5]\}, \\ \mathcal{M}(X_1|x_2) &= \text{CH}\{[1, 0], [0.4, 0.6]\}, \\ \mathcal{M}(X_1|\bar{x}_2) &= \text{CH}\{[0.25, 0.75], [0.6, 0.4]\}. \end{aligned}$$

The strong extension of these credal sets is

$$\begin{aligned} \tilde{\mathcal{M}}_1(X_1X_2) = \text{CH}\{&[0.2, 0.2, 0, 0.6], [0.2, 0.48, 0, 0.32], \\ &[0.08, 0.2, 0.12, 0.6], [0.08, 0.48, 0.12, 0.32], \\ &[0.5, 0.125, 0, 0.375], [0.5, 0.3, 0, 0.2], \\ &[0.2, 0.125, 0.3, 0.375], [0.2, 0.3, 0.3, 0.2]\}. \end{aligned}$$

which is less precise than the original model. $\diamondsuit$

It can be viewed as an advantage of compositional models that they are based on "local knowledge" even in cases when the credal network is not separately specified.

4.3 From Perfect Sequence to Credal Network

In this subsection we will present an algorithm for transforming a perfect sequence of credal sets to a credal network and we will illustrate its application on a simple example.

Having a perfect sequence $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$ ($\mathcal{M}_\ell$ being a credal set describing X_{K_ℓ}), we first order all of the variables for which at least one of the credal sets $\mathcal{M}_\ell$ is defined in such a way that first we order (in an arbitrary way) variables for which $\mathcal{M}_1$ is defined, then variables from $\mathcal{M}_2$ that are not contained in $\mathcal{M}_1$, etc. Finally we have

$$\{X_1, X_2, X_3, \ldots, X_n\} = \{X_i\}_{i \in K_1 \cup \ldots \cup K_m}.$$

Then we get a graph of the constructed evidential network in the following way:

(i) the nodes are all the variables $X_1, X_2, X_3, \ldots, X_n$;

(ii) there is an edge ($X_i \to X_j$) if there exists a credal set $\mathcal{M}_\ell$ such that both $i, j \in K_\ell$, $j \notin K_1 \cup \ldots \cup K_{\ell-1}$ and either $i \in K_1 \cup \ldots \cup K_{\ell-1}$ or $i < j$.

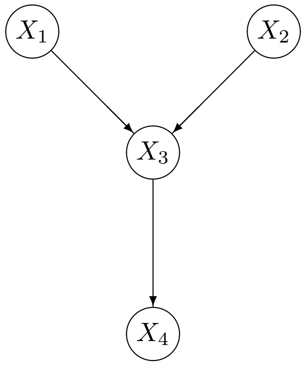

Figure 1: Graph of credal network generated from a perfect sequence

Having the structure of the credal network, i.e., graph $\mathcal{G}$, one can obtain the systems of conditional probability distributions from corresponding perfect sequences of probability distributions.

Evidently, for each j the requirement $j \in K_\ell$, $j \notin K_1 \cup \ldots \cup K_{\ell-1}$ is met exactly for one $\ell \in \{1, \ldots, n\}$. It means that all the parents of node X_j must be from the respective set $\{X_i\}_{i \in K_\ell}$ and therefore the necessary conditional probability distributions $P^i(X_j|pa(X_j))$ can easily be computed from probability distribution P^i_ℓ.

Example 3 *(Continued)* From perfect sequence

$$\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4,$$

we get the following ordering of variables

$$X_1, X_2, X_3, X_4$$

and the structure of the credal network as suggested in Figure 1. From six perfect sequences of probability distributions (12) one gets six systems of conditional probability distributions:

$$\begin{aligned} &P^1_1(X_1), P^1_2(X_2), P_3(X_3|X_1X_2), P^1_4(X_4|X_3), \\ &P^1_1(X_1), P^1_2(X_2), P_3(X_3|X_1X_2), P^2_4(X_4|X_3), \end{aligned}$$

$$P_1^1(X_1), P_2^2(X_2), P_3(X_3|X_1X_2), P_4^1(X_4|X_3),$$
$$P_1^1(X_1), P_2^2(X_2), P_3(X_3|X_1X_2), P_4^2(X_4|X_3),$$
$$P_1^2(X_1), P_2^2(X_2), P_3(X_3|X_1X_2), P_4^1(X_4|X_3),$$
$$P_1^2(X_1), P_2^2(X_2), P_3(X_3|X_1X_2), P_4^2(X_4|X_3),$$

where

$$P_1^1(X_1 = x_1) = 0.2, \qquad P_1^2(X_1 = x_1) = 0.5,$$
$$P_2^1(X_2 = x_2) = 0.5, \qquad P_2^2(X_2 = x_2) = 0.8,$$
$$P_3(X_3 = x_3|X_1 = x_1, X_2 = x_2) = 1,$$
$$P_3(X_3 = x_3|X_1 = x_1, X_2 = \bar{x}_2) = 0.75,$$
$$P_3(X_3 = x_3|X_1 = \bar{x}_1, X_2 = x_2) = 0.5,$$
$$P_3(X_3 = x_3|X_1 = \bar{x}_1, X_2 = \bar{x}_2) = 0.25,$$
$$P_4^1(X_4 = x_4|X_3 = x_3) = 0.8,$$
$$P_4^1(X_4 = x_4|X_3 = \bar{x}_3) = 0.4,$$
$$P_4^2(X_4 = x_4|X_3 = x_3) = 0.4,$$
$$P_4^2(X_4 = x_4|X_3 = \bar{x}_3) = 0.2.$$

The resulting model is again a credal set (11). ♢

5 Conclusions

This paper is devoted to the further development of the operator of composition for credal sets. Our main attention is paid to the relationship between so-called perfect sequences of credal sets, and those of (precise) probability distributions with the aim to find the relationship between credal compositional models and credal networks. We have proved that a perfect sequence of credal sets is a convex hull of perfect sequences of extreme points of these credal sets. We have also proved that perfect sequences of credal sets form a proper subclass of credal networks and, simultaneously, they are a proper superclass of separately specified credal networks. In other words, any separately specified credal network can be expressed in the form of credal compositional models and any perfect sequence of credal sets can be expressed as a credal network.

From the results presented in this paper it is evident that compositional models for credal sets can be seen as an alternative to credal networks. Therefore it seems desirable to further develop the composition operator within this framework. The first, and most important, task will be a definition of composition in the general case.

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References

[1] A. Antonucci and M. Zaffalon, Decision-theoretic specification of credal networks: a unified language for uncertain modeling with sets of Bayesian networks. *Int. J. Approx. Reasoning,* **49** (2008), 345–361.

[2] S. Benferhat, D. Dubois, L. Gracia and H. Prade, Directed possibilistic graphs and possibilistic logic. In: B. Bouchon-Meunier, R.R. Yager, (eds.) *Proc. of the 7th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-based Systems IPMU'98*, Editions E.D.K. Paris, pp. 1470–1477.

[3] B. Ben Yaghlane, Ph. Smets and K. Mellouli, Directed evidential networks with conditional belief functions. *Proceedings of ECSQARU 2003,* eds. T. D. Nielsen, N. L. Zhang, 291–305.

[4] I. Couso, S. Moral and P. Walley, Examples of independence for imprecise probabilities, *Proceedings of ISIPTA'99,* eds. G. de Cooman, F. G. Cozman, S. Moral, P. Walley, Ghent, 1999, pp. 121–130.

[5] F. G. Cozman, Sets of probability distributions, independence, and convexity, *Synthese*, **186** (2012), pp. 577–600.

[6] R. Jiroušek, Composition of probability measures on finite spaces. *Proc. of UAI'97*, (D. Geiger and P. P. Shenoy, eds.). Morgan Kaufmann Publ., San Francisco, California, pp. 274–281, 1997.

[7] R. Jiroušek, Graph modelling without graphs. *Proc. of IPMU'98*, (B. Bouchon-Meunier, R.R. Yager, eds.). Editions E.D.K. Paris, pp. 809–816, 1988.

[8] R. Jiroušek, Marginalization in composed probabilistic models. *Proc. of UAI'00* (C. Boutilier and M. Goldszmidt eds.), Morgan Kaufmann Publ., San Francisco, California, pp. 301–308, 2000.

[9] R. Jiroušek and J. Vejnarová, Construction of multidimensional models by operators of composition: current state of art. *Soft Computing,* **7** (2003), pp. 328–335.

[10] R. Jiroušek, J. Vejnarová and M. Daniel, Compositional models for belief functions. *Proceedings of 5th International Symposium on Imprecise Probability: Theories and Applications ISIPTA'07*, eds. G. De Cooman, J. Vejnarová, M. Zaffalon, Praha, 2007, pp. 243–252.

[11] R. Jiroušek and J. Vejnarová, Compositional models and conditional independence in Evidence Theory, *Int. J. Approx. Reasoning,* **52** (2011), pp. 316–334.

[12] S. Moral and A. Cano, Strong conditional independence for credal sets, *Ann. of Math. and Artif. Intell.*, **35** (2002), pp. 295–321.

[13] J. Vejnarová, Composition of possibility measures on finite spaces: preliminary results. In: *Proc. of 7th International Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems IPMU'98,* (B. Bouchon-Meunier, R.R. Yager, eds.). Editions E.D.K. Paris, 1998, pp. 25–30.

[14] J. Vejnarová, On possibilistic marginal problem, *Kybernetika* **43**, 5 (2007), pp. 657–674.

[15] J. Vejnarová, Evidential networks from a different perspective. In: *Synergies of Soft Computing and Statistics for Intelligent Data Analysis,* Soft Methods In Probability and Statistics, (2012). pp. 429–436.

[16] J. Vejnarová, Operator of composition for credal sets, *ISIPTA'13: 8th International Symposium on Imprecise Probability: Theories and Applications*, pp. 355–364.