A Logic with Upper and Lower Probability Operators

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Abstract

We present a propositional logic with unary operators that speak about upper and lower probabilities. We describe the corresponding class of models and discuss decidability issues. We provide an infinitary axiomatization for the logic and we prove that the axiomatization is sound and strongly complete. For some restrictions of the logic we provide finitary axiomatic systems.

Keywords. Probabilistic Logic, Upper and Lower Probabilities, Axiomatization, Completeness theorem.

1 Introduction

During the last few decades, uncertain reasoning has emerged as one of main fields in computer science and artificial intelligence. Many different tools are developed for representing, and reasoning with, uncertain knowledge. One particular line of research concerns the formalization in terms of probabilistic logic. After Nilsson [24] gave a procedure for probabilistic entailment which, given probabilities of premises, calculates bounds on the probabilities of the derived sentences, researchers from the field started investigation about formal systems for probabilistic reasoning [6, 7, 8, 9, 10, 13, 21, 25, 26].

However, in many applications, sharp numerical probabilities appear too simple for modelling uncertainty. This calls for developing different imprecise probability models [4, 5, 19, 22, 28, 29, 30, 32]. In order to model some situations of interest, some approaches use sets of probability measures instead of one fixed measure, and the uncertainty is represented by two boundaries – lower and upper probabilities [12, 18]. Consider the following example, essentially taken from [11].

Example 1 *Suppose that a bag contains 10 marbles and we know that 4 of them are red, and the remaining 6 are either black or green, but we do not know the exact proportion (for example, it is possible that there are no green marbles at all). The goal is to model a situation where the person picks a marble from the bag randomly. The cases when person picks up a red marble (red event), when person picks up a black marble (black event) and when person picks up a green marble (green event) will be denoted by R, B and G, respectively. Clearly, the probability of the red event is 0.4, but we cannot assign strict probability to black or green event. Therefore, we use the set of probability measures $P = \{\mu_\alpha \mid \alpha \in [0, 0.6]\}$, where μ_α assigns 0.4 probability to red event, α to black event, and $0.6 - \alpha$ to green event. We assign two functions to arbitrary set of probability measures ,P, first one is $P^\star(X) = sup\{\mu(X) \mid \mu \in P\}$ and the second one is $P_\star(X) = inf\{\mu(X) \mid \mu \in P\}$ which will be used to define a range of probabilities, i.e. they will be an upper and a lower probability, respectively.*

Halpern and Pucella [11] provided a finitary axiomatization for reasoning about linear combinations of upper probabilities, but they proved only weak completeness (every consistent formula is satisfiable). Their formulas are Boolean combinations of the expressions of the form $r_1\ell(\alpha_1) + \cdots + r_n\ell(\alpha_n) \geq r_{n+1}$, where ℓ is the upper probability operator and r_i are real numbers[1], for $i \in \{1, 2, \ldots, n+1\}$. Since nonrestricted real-valued formalisms are rich enough to express the type of a proper infinitesimal $\{0 < x < \frac{1}{n} \mid n = 1, 2, 3, \ldots\}$ (see Example 3), the logic from [11] is not compact. As an unpleasant logical consequence, for any finitary axiomatic system, there are consistent sets of formulas which are unsatisfiable [31].

In this paper, we propose sound and strongly complete (every consistent set of formulas is satisfiable) propositional logic for reasoning about lower and up-

[1] In [11], Halpern and Pucella define the rich language with formulas with all the reals as coefficients. But, in order to obtain decidability result, they have to restrict their language and allow only integer coefficients, i.e. $r_i \in \mathbb{Z}$.

per probabilities ($LUPP$[2]), whose syntax is simpler than the one in [11]. We extend propositional calculus with modal-like unary operators of the form $U_{\geq s}$ and $L_{\geq s}$, where s ranges over the unit interval of rational numbers. The intended meanings of $U_{\geq s}\alpha$ and $L_{\geq s}\alpha$ are "the upper/lower probability of α is at least s". The corresponding semantics consists of special types of Kripke models (possible worlds), with addition of sets of probability measures defined over the worlds. In order to obtain strong completeness, we use infinitary inference rules. Thus our languages are countable and formulas are finite, while only proofs are allowed to be infinite. We also propose the restricted logics $LUPP^{Fr(n)}$ (for each n in $\mathbb{N}$). For those logics, we achieve compactness using only a finite set of probability values, which is still enough for many practical applications. We propose finitary axiomatization for $LUPP^{Fr(n)}$.

From the technical point of view, we have modified some of our earlier developed completion methods presented in [14, 16, 17, 25, 27]. The complete axiomatic system for the logic is the key issue in formalizing the reasoning about lower and upper probabilities, since having a completeness theorem is the only formal way to prove the correctness of the hardware and software.

The contents of this paper are as follows. In Section 2 we recall the notions of lower and upper probability, as well as the representation theorem we use in our axiomatization. In Section 3 we present the syntax and semantics of $LUPP$ and discuss its decidability. In Section 4 we propose an axiomatization for the logic, and we prove some auxiliary propositions. We prove the soundness and completeness of the axiomatization in Section 5. In Section 6 we present the logics $LUPP^{Fr(n)}$, where the probabilities are restricted to a finite set. We conclude in Section 7.

2 Preliminaries

Let $W \neq \emptyset$ and let H be an algebra of subsets of W, i.e., a set of subsets of W such that:

- $W \in H$,
- if $A, B \in H$, then $W \setminus A \in H$ and $A \cup B \in H$.

A function $\mu : H \longrightarrow [0,1]$ is a finitely additive probability measure, if the following conditions hold:

- $\mu(W) = 1$,
- $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$.

For a set P of probability measures defined on H, the lower probability measure $P_\star$ and the upper probability measure $P^\star$ are defined by

- $P_\star(X) = \inf\{\mu(X) \mid \mu \in P\}$
- $P^\star(X) = \sup\{\mu(X) \mid \mu \in P\}$

for every $X \in H$. In the proof of soundness and completeness, we will use the following basic properties of $P_\star$ and $P^\star$:

- $P_\star(X) \leq P^\star(X)$,
- $P_\star(X) = 1 - P^\star(X^c)$,
- $P^\star(X \cup Y) \leq P^\star(X) + P^\star(Y)$, whenever $X \cap Y = \emptyset$.

In order to axiomatize upper and lower probabilities, we need to completely characterize $P_\star$ and $P^\star$ with a finite number of properties. Many complete characterizations are proposed in the literature, the earliest appears to be by Lorentz [20]. We will use the characterization by Anger and Lembcke [2] (also used by Halpern and Pucella [11, Theorem 2.3]). We start with the definition of (n,k)-cover.

Definition 1 ((n,k)-cover) . *A set A is said to be covered n times by a multiset $\{\{A_1,\dots,A_m\}\}$ of sets if every element of A appears in at least n sets from $A_1,\dots,A_m$, i.e., for all $x \in A$, there exists $i_1,\dots,i_n$ in $\{1,\dots,m\}$ such that for all $j \leq n$, $x \in A_{i_j}$. An (n,k)-cover of (A,W) is a multiset $\{\{A_1,\dots,A_m\}\}$ that covers W k times and covers A $n+k$ times.*

Theorem 1 (Anger and Lembcke [2]) *Let W be a set, H an algebra of subsets of W, and f a function $f : H \longrightarrow [0,1]$. There exists a set P of probability measures such that $f = P^\star$ iff f satisfies the following three properties:*

(1) $f(\emptyset) = 0$,

(2) $f(W) = 1$,

(3) *for all natural numbers m,n,k and all subsets $A_1,\dots,A_m$ in H, if $\{\{A_1,\dots,A_m\}\}$ is an (n,k)-cover of (A,W), then $k + nf(A) \leq \sum_{i=1}^{m} f(A_i)$.*

3 The Logic $LUPP$

In this section we will describe the syntax and semantics of the logic $LUPP$, and we discuss the decidability problem of satisfiability of $LUPP$-formulas.

3.1 Syntax

Let S be the set of rational numbers from $[0,1]$ and let $\mathcal{L} = \{p,q,r,\dots\}$ be a countable set of propositional letters. The language of logic $LUPP$ consists of the elements of set $\mathcal{L}$, classical propositional connectives $\neg$ and $\wedge$ and the lists of upper probability operators $U_{\geq s}$ and $L_{\geq s}$, for every $s \in S$. The set of all classical propositional formulas over $\mathcal{L}$ is defined as usual,

[2] LUP stands for "lower and upper probability", while the second P indicates that the logic is propositional.

and we will denote it by For_C. We will denote the propositional formulas by α, β and γ.

Definition 2 ***(Lower and upper probabilistic formulas)*** *If $\alpha \in For_C$ and $s \in S$, then a basic lower probability formula is any formula of the form $L_{\geq s}\alpha$, and a basic upper probability formula is any formula of the form $U_{\geq s}\alpha$. The set of all lower and upper probabilistic formulas, denoted by For_P, is the smallest set containing all basic lower and upper probability formulas which is closed under Boolean connectives.*

We denote the lower and upper probabilistic formulas by ϕ and ψ, possibly indexed. Let

$$For_{LUPP} = For_C \cup For_P.$$

The formulas from the set For_{LUPP} will be denoted by ρ and σ, possibly with subscripts.

We use the following abbreviations to introduce other types of inequalities: $U_{<s}\alpha$ is $\neg U_{\geq s}\alpha$, $L_{<s}\alpha$ is $\neg L_{\geq s}\alpha$, $U_{\leq s}\alpha$ is $L_{\geq 1-s}\neg\alpha$, $L_{\leq s}\alpha$ is $U_{\geq 1-s}\neg\alpha$, $U_{=s}\alpha$ is $U_{\leq s}\alpha \wedge U_{\geq s}\alpha$, $L_{=s}\alpha$ is $L_{\leq s}\alpha \wedge L_{\geq s}\alpha$, $U_{>s}\alpha$ is $\neg U_{\leq s}\alpha$, $L_{>s}\alpha$ is $\neg L_{\leq s}\alpha$. We also denote both $\alpha \wedge \neg\alpha$ and $\phi \wedge \neg\phi$ by $\perp$ (and similarly for $\top$).

Note that formulas are defined in the same style as in [3, 26], i.e. neither mixing of pure propositional formulas and lower and upper probabilistic formulas, nor nested lower and upper probability operators is allowed.

Example 2 *Continuing Example 1, it is clear that upper and lower probability, for the case that picked marble is green or black, are equal to* 0.6. *If there are no green marbles at all, then we obtain that lower probability for the case that picked marble is not green equals to* 1. *We can express that by the following formula of our language:*

$$U_{=0.6}(G \vee B) \wedge L_{=0.6}(G \vee B) \Rightarrow L_{=1}\neg G.$$

Another example of a lower and upper probabilistic formula is

$$U_{<\frac{1}{3}}\alpha \rightarrow L_{\geq\frac{1}{2}}(\alpha \wedge \beta),$$

where α, $\beta \in For_C$.

Next we state two formulas that are not well defined lower and upper probabilistic formulas of the logic $LUPP$:

$$\alpha \wedge U_{=1}\beta, \qquad U_{\geq s}U_{\geq r}\alpha.$$

The first formula is not well defined since it is a Boolean combination of pure propositional formula and an upper probabilistic formula, while the second formula is not well defined lower and upper probabilistic formula because it contains nested operators.

3.2 Semantics

The semantics for $LUPP$ is based on the possible-world approach.

Definition 3 ($LUPP$-structure) *An $LUPP$-structure is a tuple $\langle W, H, P, v\rangle$, where:*

- *W is a nonempty set of worlds.*
- *H is an algebra of subsets of W. The elements of H are called measurable worlds.*
- *P is a set of finitely additive probability measures defined on H.*
- *$v : W \times \mathcal{L} \longrightarrow \{true, false\}$ provides for each world $w \in W$ a two-valued evaluation of the primitive propositions, which is extended to classical propositional formulas as usual.*

For given $\alpha \in For_C$ and $LUPP$-structure M, let $[\alpha]_M = \{w \in W \mid v(w)(\alpha) = true\}$. We will not write the subscript M when it's clear from context.

Definition 4 (Measurable structure) *The structure M is measurable if $[\alpha]_M \in H$ for every $\alpha \in For_C$. The class of a measurable structures of the logic $LUPP$ will be denoted by $LUPP_{Meas}$.*

Definition 5 (Satisfiability relation) *The satisfiability relation $\models \subseteq LUPP_{Meas} \times For_{LUPP}$ is defined in the following way:*

- *$M \models \alpha$ iff $v(w)(\alpha) = true$, for all $w \in W$,*
- *$M \models U_{\geq s}\alpha$ iff $P^\star([\alpha]) \geq s$,*
- *$M \models L_{\geq s}\alpha$ iff $P_\star([\alpha]) \geq s$,*
- *$M \models \neg\phi$ iff it is not the case that $M \models \phi$,*
- *$M \models \phi \wedge \psi$ iff $M \models \phi$ and $M \models \psi$.*

Definition 6 (Satisfiability of a formula) *A formula $\rho \in For_{LUPP}$ is satisfiable if there is an $LUPP_{Meas}$-model M such that $M \models \rho$; ρ is valid if for every $LUPP_{Meas}$-model M, $M \models \rho$. A set of formulas T is satisfiable if there is an $LUPP_{Meas}$-model M such that $M \models \rho$ for every $\rho \in T$.*

Example 3 *Consider the set $T = \{\neg U_{=0}\alpha\} \cup \{U_{<\frac{1}{n}}\alpha \mid n$ is a positive integer$\}$. Every finite subset of T is $LUPP_{Meas}$-satisfiable, but the set T itself is not. Therefore, the compactness theorem which states that "if every finite subset of T is satisfiable, then T is satisfiable" does not hold for $LUPP$.*

3.3 Decidability

Recall that Halpern and Pucella [11] provide decidability result for the formulas which are Boolean combinations of the expressions of the form

$$r_1\ell(\alpha_1) + \cdots + r_n\ell(\alpha_n) \geq r_{n+1},$$

where ℓ is the upper probability operator and r_i are integers, for $i \in \{1, 2, \ldots, n+1\}$. First of all, note that using only integers as coefficients has the same expressive power as using all rational numbers. For example, if we want to express $\frac{3}{7}\ell(\alpha) \geq 1$ by using only integers, we can reformulate the formula as $3\ell(\alpha) \geq 7$, etc. Also note that our formula $U_{\geq s}\alpha$ is satisfiable iff the formula $\ell(\alpha) \geq s$ is satisfiable in the logic from [11]. Similarly, $L_{\geq s}\alpha$ is satisfiable iff the formula $-\ell(\neg\alpha) \geq -(1-s)$ is satisfiable. Then decidability of our logic is a consequence of decidability of the logic from [11]. Moreover, since the problem of deciding whether a formula of their language is satisfiable is NP-complete [11, Theorem 5.2], we have an upper bound of the decidability problem for $LUPP$. The lower bound follows from the fact that the complexity of decision problem for classical propositional logic is NP-complete. Thus, the satisfiability problem for $LUPP$-formulas is NP-complete as well.

4 The Axiomatization Ax_{LUPP}

We will introduce an axiomatic system for the logic $LUPP$ which will be denoted by Ax_{LUPP}.

Axiom schemes

(1) all instances of the classical propositional tautologies

(2) $U_{\leq 1}\alpha \wedge L_{\leq 1}\alpha$

(3) $U_{\leq r}\alpha \to U_{<s}\alpha$, $s > r$

(4) $U_{<s}\alpha \to U_{\leq s}\alpha$

(5) $(U_{\leq r_1}\alpha_1 \wedge \cdots \wedge U_{\leq r_m}\alpha_m) \to U_{\leq r}\alpha$, if $\alpha \to \bigvee_{J\subseteq\{1,\ldots,m\},|J|=k+n}\bigwedge_{j\in J}\alpha_j$ and $\bigvee_{J\subseteq\{1,\ldots,m\},|J|=k}\bigwedge_{j\in J}\alpha_j$ are propositional tautologies, where $r = \frac{\sum_{i=1}^m r_i - k}{n}$, $n \neq 0$

(6) $\neg(U_{\leq r_1}\alpha_1 \wedge \cdots \wedge U_{\leq r_m}\alpha_m)$, if $\bigvee_{J\subseteq\{1,\ldots,m\},|J|=k}\bigwedge_{j\in J}\alpha_j$ is a propositional tautology and $\sum_{i=1}^m r_i < k$

(7) $L_{=1}(\alpha \to \beta) \to (U_{\geq s}\alpha \to U_{\geq s}\beta)$

Inference Rules

(1) From ρ and $\rho \to \sigma$ infer σ

(2) From α infer $L_{\geq 1}\alpha$

(3) From the set of premises

$$\{\phi \to U_{\geq s-\frac{1}{k}}\alpha \mid k \geq \frac{1}{s}\}$$

infer $\phi \to U_{\geq s}\alpha$

(4) From the set of premises

$$\{\phi \to L_{\geq s-\frac{1}{k}}\alpha \mid k \geq \frac{1}{s}\}$$

infer $\phi \to L_{\geq s}\alpha$.

We have, by Axiom 1, that the classical propositional logic is sublogic of $LUPP$. Axiom 2 announce that the upper bound for upper and lower probabilities is 1. Axioms 5 and 6 are the logical analogue of the third condition from Theorem 1. To see that, note that equivalent way to say that $\{\{A_1, \ldots, A_m\}\}$ covers a set A n times is that

$$A \subseteq \bigcup_{J\subseteq\{1,\ldots,m\},|J|=n}\ \bigcap_{j\in J} A_j.$$

Therefore, the condition that the formula $\alpha \to \bigvee_{J\subseteq\{1,\ldots,m\},|J|=k+n}\bigwedge_{j\in J}\alpha_j$ is a tautology gives us that $[\alpha]$ is covered $n+k$ times by a multiset $\{\{[\alpha_1], \ldots, [\alpha_m]\}\}$, while the condition that $\bigvee_{J\subseteq\{1,\ldots,m\},|J|=k}\bigwedge_{j\in J}\alpha_j$ is a propositional tautology ensures that $W = [\top]$ is covered k times by a multiset $\{\{[\alpha_1], \ldots, [\alpha_m]\}\}$. Axiom 7 is crucial for proving that equivalent formulas have equal lower and upper probabilities.

Rule 1 is modus ponens, Rule 2 is the lower probability necessitation. Both Rule 3 and Rule 4 are infinitary rules of inference and Rule 3 intuitively says that if upper probability is arbitrary close to s then it is at least s, while Rule 4 intuitively says that if lower probability is arbitrary close to s then it is at least s.

Definition 7 (Inference relation)

- *$T \vdash \rho$ (ρ is derivable from T) if there is an at most denumerable sequence of formulas $\rho_1, \rho_2, \ldots, \rho$, such that every ρ_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule;*

- *$\vdash \rho$ (ρ is a theorem) iff $\emptyset \vdash \rho$;*

- *T is consistent if there is at least a formula $\alpha \in For_C$ and a formula $\phi \in For_P$ that are not deducible from T, otherwise T is inconsistent;*

- *T is maximally consistent set if it is consistent and:*

(1) *for every* $\alpha \in For_C$, *if* $T \vdash \alpha$, *then* $\alpha \in T$ *and* $L_{\geq 1}\alpha \in T$

(2) *for every* $\phi \in For_P$, *either* $\phi \in T$ *or* $\neg\phi \in T$.

- T *is deductively closed if for every* $\rho \in For_{LUPP}$, *if* $T \vdash \rho$, *then* $\rho \in T$.

The equivalent way to say that T is inconsistent is that $T \vdash \bot$. Note that it is not required that for every $\alpha \in For_C$, either α or $\neg\alpha$ belongs to a maximal consistent set (as it is done for formulas from For_P). Otherwise, by Rule 2, for each α we would have $L_{\geq 1}\alpha$ or $L_{\geq 1}\neg\alpha$.

Theorem 2 (Deduction theorem) *Let* T *be a set of formulas. Then* $T \cup \{\phi\} \vdash \psi$ *iff* $T \vdash \phi \to \psi$.

Proof. The only interesting case is when $\phi,\psi \in For_P$.
($\Leftarrow$) Direct consequence of Rule 1.
($\Rightarrow$) Suppose that $T \cup \{\phi\} \vdash \psi$. We will use the induction on the length of the inference.
The cases when either $\vdash \psi$ or $\phi = \psi$ or ψ is obtained by application of modus ponens are the same as in the classical propositional case. Thus, let us consider the case where $\psi = L_{\geq 1}\alpha$ is obtained from $T \cup \{\phi\}$ by an application of Rule 2. In that case:

- $T, \phi \vdash \alpha$
- $T, \phi \vdash L_{\geq 1}\alpha$ by Rule 2

However, since $\alpha \in For_C$ and $\phi \in For_P$, ϕ cannot affect the proof of α from $T \cup \{\phi\}$, and we have:

(1) $T \vdash \alpha$

(2) $T \vdash L_{\geq 1}\alpha$ by Rule 2

(3) $T \vdash L_{\geq 1}\alpha \to (\phi \to L_{\geq 1}\alpha)$

(4) $T \vdash \phi \to L_{\geq 1}\alpha$ by Rule 1.

Next, let us consider the case where $\psi = \psi_1 \to U_{\geq s}\alpha$ is obtained from $T \cup \{\phi\}$ by an application of Rule 3. Then:

(1) $T, \phi \vdash \psi_1 \to U_{\geq s-\frac{1}{k}}\alpha$, for all $k \geq \frac{1}{s}$

(2) $T \vdash \phi \to (\psi_1 \to U_{\geq s-\frac{1}{k}}\alpha)$, by the induction hypothesis

(3) $T \vdash (\phi \wedge \psi_1) \to U_{\geq s-\frac{1}{k}}\alpha$

(4) $T \vdash (\phi \wedge \psi_1) \to U_{\geq s}\alpha$, by Rule 3

(5) $T \vdash \phi \to \psi$.

If the formula is obtained by an application of Rule 4, the proof is similar. □

We will not always explicitly emphasize moments in proofs where we use Deduction theorem.

Lemma 1 $\vdash U_{\leq r}\alpha \to L_{\leq r}\alpha$.

Proof. We consider two cases.

(1) $r \neq 1$. From Axiom (6) we obtain that $\neg(U_{\leq r}\alpha \wedge U_{\leq s}\neg\alpha)$, whenever $r + s < 1$. Therefore $U_{\leq r}\alpha \to U_{>s}\neg\alpha$, and because that holds for every $s < 1-r$, by inference rule (3) we have $U_{\leq r}\alpha \to U_{\geq 1-r}\neg\alpha$, i.e. $U_{\leq r}\alpha \to L_{\leq r}\alpha$.

(2) $r = 1$. Direct consequence of Axiom (2). □

Consequently, we obtain that $\vdash L_{\geq r}\alpha \to U_{\geq r}\alpha$, for each $r \in S$.

Lemma 2

(a) $\vdash U_{\geq 0}\alpha$

(b) $\alpha \vdash U_{=1}\alpha$

(c) $\vdash U_{=1}\top$

(d) $\vdash U_{=0}\bot$

(e) $\vdash U_{\geq s}\alpha \to U_{>r}\alpha$, $s > r$

(f) $\vdash U_{>s}\alpha \to U_{\geq s}\alpha$

(g) If $T \vdash \alpha \leftrightarrow \beta$ *then* $T \vdash U_{\geq s}\alpha \leftrightarrow U_{\geq s}\beta$

Proof.
(a) From Axiom (2), considering $\neg\alpha$, we have that $\vdash L_{\geq 0}\alpha$, and therefore, by Lemma 1 we have that $\vdash U_{\geq 0}\alpha$.
(b) Direct consequence of Inference Rule 2 and Lemma 1. The proofs of (c) and (d) are straightforward, (e) and (f) are obtained from Axioms (3) and (4) and contraposition, and (g) is direct consequence of Rule (2) and Axiom (7). □

5 Soundness and Completeness

5.1 Soundness

Theorem 3 (Soundness) *The axiomatic system* Ax_{LUPP} *is sound with respect to the class of* $LUPP_{Meas}$*-models.*

Proof. Our goal is to show that every instance of an axiom schemata holds in every model and that the inference rules preserve the validity. For example, let us consider Axiom 5. Suppose that $\alpha \rightarrow \bigvee_{J\subseteq\{1,\dots,m\},|J|=k+n} \bigwedge_{j\in J} \alpha_j$ and $\bigvee_{J\subseteq\{1,\dots,m\},|J|=k} \bigwedge_{j\in J} \alpha_j$ are propositional tautologies, and suppose that $(U_{\leq r_1}\alpha_1 \wedge \dots \wedge U_{\leq r_m}\alpha_m)$ holds in a model $M = \langle W, H, P, v\rangle$. We already explained that this means that a multiset $\{\{[\alpha_1],\dots,[\alpha_m]\}\}$ is an (n,k)-cover of $([\alpha],[\top])$. Also, the inequalities $P^\star([\alpha_1]) \leq r_1, \dots, P^\star([\alpha_m]) \leq r_m$ hold, by assumption. Since $P^\star$ is an upper probability measure, by Theorem 1, we know that $k + nP^\star([\alpha]) \leq \sum_{i=1}^m P^\star([\alpha_i])$, so we obtain that $P^\star([\alpha]) \leq \frac{\sum_{i=1}^m r_i - k}{n}$, $n \neq 0$ therefore $P^\star([\alpha]) \leq r$, where $r = \frac{\sum_{i=1}^m r_i - k}{n}$, i.e. $M \models U_{\leq r}\alpha$ as well. Consider now the Axiom (7). If $M \models L_{=1}(\alpha \rightarrow \beta)$, we have that $P_\star([\alpha \rightarrow \beta]) = 1$, and therefore $P^\star([\alpha \wedge \neg\beta]) = 1 - P_\star([\alpha \rightarrow \beta]) = 0$. Therefore $P^\star([\alpha]) = P^\star([\alpha \wedge \beta] \cup [\alpha \wedge \neg\beta]) \leq P^\star([\alpha \wedge \beta]) + P^\star([\alpha \wedge \neg\beta]) \leq P^\star([\beta])$. Hence, if $P^\star([\alpha]) \geq s$, then $P^\star([\beta]) \geq s$, so $M \models U_{\geq s}\alpha \rightarrow U_{\geq s}\beta$. The other axioms can be proved to be valid in a similar way and the proof is easier.

Rule (1) is validity-preserving for the same reason as in classical logic. Rule (2): if α holds in $M = \langle W, H, P, v\rangle$, then $[\alpha] = W$, and therefore $\mu([\alpha]) = 1$ for every $\mu \in P$. Then $P_\star([\alpha]) = 1$, so $M \models L_{\geq 1}\alpha$. Rule (3): Suppose that $M \models \phi \rightarrow U_{\geq s-\frac{1}{k}}\alpha$ whenever $k \geq \frac{1}{s}$. If $M \not\models \phi$, then obviously $M \models \phi \rightarrow U_{\geq s}\alpha$. Otherwise $M \models U_{\geq s-\frac{1}{k}}\alpha$ for every $k \geq \frac{1}{s}$, so $M \models U_{\geq s}\alpha$ because of the properties of the set of reals. Rule (4) is validity-preserving for the same reason as Rule (3). □

5.2 Completeness

In order to prove the completeness theorem we start with some auxiliary statements. After that, we show how to extend a consistent set of formulas T to a maximal consistent set of formulas $T^\star$. Finally, we construct the canonical model using the set $T^\star$ such that $M_{T^\star} \models \rho$ iff $\rho \in T^\star$.

Lemma 3 *Let T be a consistent set of formulas.*

(1) For any formula $\phi \in For_P$, either $T \cup \{\phi\}$ is consistent or $T \cup \{\neg\phi\}$ is consistent.

(2) If $\neg(\phi \rightarrow U_{\geq s}\alpha) \in T$, then there is some $n > \frac{1}{s}$ such that $T \cup \{\phi \rightarrow \neg U_{\geq s-\frac{1}{n}}\alpha\}$ is consistent.

(3) If $\neg(\phi \rightarrow L_{\geq s}\alpha) \in T$, then there is some $n > \frac{1}{s}$ such that $T \cup \{\phi \rightarrow \neg L_{\geq s-\frac{1}{n}}\alpha\}$ is consistent.

Proof.

(1) If $T \cup \{\phi\} \vdash \perp$, and $T \cup \{\neg\phi\} \vdash \perp$, then by Deduction theorem we have $T \vdash \neg\phi$ and $T \vdash \phi$. Contradiction.

(2) Suppose that for all $n > \frac{1}{s}$:

$$T, \phi \rightarrow \neg U_{\geq s-\frac{1}{n}}\alpha \vdash \perp.$$

Therefore, by Deduction theorem and propositional reasoning, we have

$$T \vdash \phi \rightarrow U_{\geq s-\frac{1}{n}}\alpha,$$

and by application of Rule 3 we obtain $T \vdash \phi \rightarrow U_{\geq s}\alpha$. Contradiction with the fact that $\neg(\phi \rightarrow U_{\geq s}\alpha) \in T$.

(3) can be proved in a similar way. □

Theorem 4 *Every consistent set can be extended to a maximal consistent set.*

Proof. Consider a consistent set T. By $Cn_C(T)$ we will denote the set of all classical formulas that are consequences of T. Let $\phi_0, \phi_1, \dots$ be an enumeration of all formulas from For_P. We define a sequence of sets T_i, $i = 0, 1, 2, \dots$ as follows:

(1) $T_0 = T \cup Cn_C(T) \cup \{L_{\geq 1}\alpha \mid \alpha \in Cn_C(T)\}$

(2) for every $i \geq 0$,

  (a) if $T_i \cup \{\phi_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\phi_i\}$, otherwise

  (b) if ϕ_i is of the form $\psi \rightarrow U_{\geq s}\beta$, then $T_{i+1} = T_i \cup \{\neg\phi_i, \psi \rightarrow \neg U_{\geq s-\frac{1}{n}}\beta\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise

  (c) if ϕ_i is of the form $\psi \rightarrow L_{\geq s}\beta$, then $T_{i+1} = T_i \cup \{\neg\phi_i, \psi \rightarrow \neg L_{\geq s-\frac{1}{n}}\beta\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise

  (d) $T_{i+1} = T_i \cup \{\neg\phi_i\}$.

(3) $T^\star = \bigcup_{i=0}^\infty T_i$.

The set T_0 is obviously consistent because it contains consequences of an consistent set. Note that existence of the natural numbers (n) from the steps $2(b)$ and $2(c)$ of the construction is provided by Lemma 3, and each T_i is consistent.

It still remains to show that $T^\star$ is maximal consistent set. The steps (1) and (2) of the above construction ensure that $T^\star$ is maximal.

$T^\star$ obviously doesn't contain all formulas. If $\alpha \in For_C$, by the construction of T_0, α and $\neg\alpha$ can not be both in T_0. For a formula $\phi \in For_P$, the set $T^\star$ does not contain both $\phi = \phi_i$ and $\neg\phi = \phi_j$, because the set $T_{\max\{i,j\}+1}$ is consistent.

Let us prove that $T^\star$ is deductively closed. If a formula $\alpha \in For_C$ and $T \vdash \alpha$, then by the construction of T_0, $\alpha \in T^\star$ and $L_{\geq 1}\alpha \in T^\star$. Let $\phi \in For_P$. It can be easily proved (induction on the length of the inference) that if $T^\star \vdash \phi$, then $\phi \in T^\star$. Note the fact that if $\phi = \phi_j$ and $T_i \vdash \phi$ it has to be $\phi \in T^\star$ because $T_{\max\{i,j\}+1}$ is consistent. Suppose that the sequence $\phi_1, \phi_2, \ldots, \phi$ is the proof of ϕ from $T^\star$. If mentioned sequence is finite, there must be some set T_i such that $T_i \vdash \phi$, and $\phi \in T^\star$. Therefore, suppose that the sequence is countably infinite. We can show that, for every i, if ϕ_i is obtained by an application of an arbitrary inference rule, and all the premises belong to $T^\star$, then, also $\phi_i \in T^\star$. If the inference rule is finitary one, then there must be a set T_j which contains all the premises and $T_j \vdash \phi_i$. So, we conclude that $\phi_i \in T^\star$. Now, consider the infinitary Rule 3. Let $\phi_i = \psi \to U_{\geq s}\alpha$ be obtained from the set of premises $\{\phi_i^k = \psi \to U_{\geq s_k}\alpha \mid s_k \in S\}$. By the induction hypothesis, we have that $\phi_i^k \in T^\star$, for every k. If $\phi_i \notin T^\star$, by step (2)(b) of the construction, there are some l and j so that $\neg(\psi \to U_{\geq s}\alpha)$, $\psi \to \neg U_{\geq s-\frac{1}{l}}\alpha \in T_j$. Thus, we have that for some $j' \geq j$:
- $\psi \wedge \neg U_{\geq s}\alpha \in T_{j'}$,
- $\psi \in T_{j'}$,
- $\neg U_{\geq s-\frac{1}{l}}\alpha$, $U_{\geq s-\frac{1}{l}}\alpha \in T_{j'}$.

Contradiction with the consistency of a set $T_{j'}$.
If we consider the infinitary Rule 4, the proof is similar.

Thus, $T^\star$ is deductively closed set which does not contain all formulas, so it is consistent. □

Definition 8 *If $T^\star$ is the maximally consistent set of formulas, then a tuple $M_{T^\star} = \langle W, H, P, \upsilon\rangle$ is defined:*

- *$W = \{w \mid w \models Cn_C(T)\}$ contains all classical propositional interpretations that satisfy the set $Cn_C(T)$,*
- *$H = \{[\alpha] \mid \alpha \in For_C\}$, where $[\alpha] = \{w \in W \mid w \models \alpha\}$,*
- *P is any set of probability measures such that $P^\star([\alpha]) = \sup\{s \mid U_{\geq s}\alpha \in T^\star\}$,*
- *for every world w and every propositional letter p, $\upsilon(w,p) = true$ iff $w \models p$.*

Lemma 4 *$M_{T^\star}$ is well defined.*

Proof. The prove that H is an algebra is straightforward.

First, $P^\star([\alpha]) := \sup\{s \mid U_{\geq s}\alpha \in T^\star\}$ is well defined because the value of the supremum does not depend on the choice of element from $[\alpha]$, by Lemma 2(g). Let's prove that $P^\star$ is an upper probability measure for some set of probability measures P. It is sufficient to prove the three conditions from Theorem 1. The conditions $P^\star(\emptyset) = 0$ and $P^\star(W) = 1$ are trivial. The only thing left to prove is that if $\{\{[\alpha_1], \ldots, [\alpha_m]\}\}$ is (n,k)-cover of $([\alpha], W)$, then $k + nP^\star([\alpha]) \leq \sum_{i=1}^m P^\star([\alpha_i])$.

Let $P^\star([\alpha_i]) = a_i$, i.e. $\sup\{r \mid U_{\geq r}\alpha_i \in T^\star\} = a_i$, $i = 1, \ldots, m$. For arbitrary $\varepsilon > 0$ there exists rational numbers $q_i \in (a_i, a_i + \varepsilon)$ such that $U_{\leq q_i}\alpha_i \in T^\star$ (otherwise $U_{>q_i}\alpha_i \in T^\star$ which is contradiction with the fact that a_i is supremum). Hence, we have $T^\star \vdash U_{\leq q_1}\alpha_1 \wedge \cdots \wedge U_{\leq q_m}\alpha_m$, and by Axiom 5, we have $T^\star \vdash U_{\leq q}\alpha$, where $q = \frac{\sum_{i=1}^m q_i - k}{n}$, $n \neq 0$ i.e., $\sup\{r \mid U_{\geq r}\alpha_i \in T^\star\} \leq q$ or $P^\star([\alpha]) \leq q$. Therefore, we have $P^\star([\alpha]) \leq \frac{\sum_{i=1}^m q_i - k}{n} = \frac{\sum_{i=1}^m a_i + m\varepsilon - k}{n}$, and because this holds for every $\varepsilon > 0$ we obtain $k + nP^\star([\alpha]) \leq \sum_{i=1}^m P^\star([\alpha_i])$.

If $n = 0$, we need to show that $k \leq \sum_{i=1}^m P^\star([\alpha_i])$. Reasoning as above, we have that $T^\star \vdash U_{\leq q_1}\alpha_1 \wedge \cdots \wedge U_{\leq q_m}\alpha_m$, for some $q_i \in (a_i, a_i + \varepsilon)$, and because of Axiom (6), how $\bigvee_{J \subseteq \{1,\ldots,m\}, |J|=k} \bigwedge_{j \in J} \alpha_j$ are propositional tautologies, we have that $\sum_{i=1}^m q_i \geq k$. Since that holds for every $\varepsilon > 0$, we obtain $\sum_{i=1}^m a_i \geq k$. □

Lemma 5 *Let $T^\star$ be a maximal consistent set of formulas. Then, $M_{T^\star} \in LUPP_{Meas}$.*

Proof. Directly from the construction of $M_{T^\star}$. □

Now we are ready to prove the main result of this paper.

Theorem 5 (Strong completeness) . *A set of formulas T is consistent iff it is $LUPP_{Meas}-satisfiable$.*

Proof. Direction from right to left follows from the Soundness Theorem. For the proof of the other direction we construct $LUPP_{Meas}$-model $M_{T^\star}$ and show that for every $\rho \in For_{LUPP}$, $M_{T^\star} \models \rho$ iff $\rho \in T^\star$. We use the induction on the complexity of the formula.

- $\rho = \alpha \in For_C$. If $\alpha \in Cn_C(T)$, then by definition of $M_{T^\star}$ we have $M_{T^\star} \models \alpha$. Conversely, if $M_{T^\star} \models \alpha$, by the completeness of classical propositional logic we have that $\alpha \in Cn_C(T)$.
- Consider the case when $\rho = U_{\geq s}\alpha$. If $U_{\geq s}\alpha \in T^\star$, then $\sup\{r \mid U_{\geq r}\alpha \in T^\star\} = P^\star([\alpha]) \geq s$, and so $M_{T^\star} \models U_{\geq s}\alpha$. Now, suppose that $M_{T^\star} \models U_{\geq s}\alpha$, i.e. $\sup\{r \mid U_{\geq r}\alpha \in T^\star\} \geq s$. If $P^\star([\alpha]) > s$, then by the properties of supremum and monotonicity of $P^\star$, we have $U_{\geq s}\alpha \in T^\star$. If $P^\star([\alpha]) = s$, then, as a direct consequence of inference Rule 3, we have that $U_{\geq s}\alpha \in T^\star$.

- Next, let $\rho = L_{\geq s}\alpha$, i.e. $\rho = U_{\leq 1-s}\neg\alpha$. First, suppose that $U_{\leq 1-s}\neg\alpha \in T^\star$. We want to show that $\sup\{r \mid U_{\geq r}\neg\alpha \in T^\star\} \leq 1-s$, so suppose that $\sup\{r \mid U_{\geq r}\neg\alpha \in T^\star\} > 1-s$. Then, there exist a rational number $q \in (1-s, 1-s+\epsilon]$, for some $\epsilon > 0$, such that $U_{\geq q}\alpha \in T^\star$. Hence, $U_{>1-s}\neg\alpha \in T^\star$ which leads us to contradiction. So, $\sup\{r \mid U_{\geq r}\neg\alpha \in T^\star\} \leq 1-s$, i.e. $P^\star([\neg\alpha]) \leq 1-s$ and thus we obtain $M_{T^\star} \models L_{\geq s}\alpha$. Now, for the other direction, suppose that $M_{T^\star} \models U_{\leq 1-s}\neg\alpha$, i.e. $\sup\{r \mid U_{\geq r}\neg\alpha \in T^\star\} \leq 1-s$. Consider the following two cases:

  (1) $\sup\{r \mid U_{\geq r}\neg\alpha \in T^\star\} < 1-s$. Then, if $U_{>1-s}\neg\alpha \in T^\star$, then also $U_{\geq 1-s}\neg\alpha \in T^\star$, so $\sup\{r \mid U_{\geq r}\neg\alpha \in T^\star\} \geq 1-s$. Contradiction.

  (2) $\sup\{r \mid U_{\geq r}\neg\alpha \in T^\star\} = 1-s$. We want to show that then must be $\inf\{r \mid U_{\leq r}\neg\alpha \in T^\star\} = 1-s$ as well. First, suppose that $\inf\{r \mid U_{\leq r}\neg\alpha \in T^\star\} < 1-s$. Hence, there exist a rational number $q_1 \in [1-s-\epsilon, 1-s)$ such that $U_{\leq q_1}\neg\alpha \in T^\star$, and so $U_{<1-s}\neg\alpha \in T^\star$, contradiction with the fact that $U_{\geq 1-s}\neg\alpha \in T^\star$ (direct consequence of inference rule (3)). Now, suppose that $\inf\{r \mid U_{\leq r}\neg\alpha \in T^\star\} > 1-s$, i.e. $\inf\{r \mid U_{\leq r}\neg\alpha \in T^\star\} = 1-s+\varepsilon$. Take an arbitrary rational number $q_2 \in (1-s, 1-s+\varepsilon)$ and then both $U_{\leq q_2}\neg\alpha \in T^\star$ and $U_{\geq q_2}\neg\alpha \in T^\star$ leads us to contradiction (because of the properties of infimum and supremum), which is impossible. Therefore, $\inf\{r \mid U_{\leq r}\neg\alpha \in T^\star\} = 1-s$, or equivalently $\inf\{r \mid L_{\geq 1-r}\alpha \in T^\star\} = 1-s$ and then, by the inference Rule 4, we obtain that $L_{\geq s}\alpha \in T^\star$.

- Now, let $\rho = \neg\psi \in For_P$. Then $M_{T^\star} \models \neg\psi$ iff it is not the case that $M_{T^\star} \models \psi$ iff $\psi \notin T^\star$ iff $\neg\psi \in T^\star$.

- Finally, let $\rho = \phi\wedge\psi \in For_P$. Then, $M_{T^\star} \models \phi\wedge\psi$ iff $M_{T^\star} \models \phi$ and $M_{T^\star} \models \psi$ iff $\phi, \psi \in T^\star$ iff $\phi \wedge \psi \in T^\star$. □

6 The Logic $LUPP^{Fr(n)}$

In this section we introduce the Logic $LUPP^{Fr(n)}$ which is similar to $LUPP$. The main difference is that the finitely additive measures map H to $N = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, for a fixed positive integer n. Therefore, we obtain countably many different logics, one for each n. Considering the semantics, a model is the tuple $\langle W, H, P, \upsilon\rangle$ defined as above but the set P consists of finitely additive measures with restricted range N, i.e., for each $\mu \in P$, $\mu : H \longrightarrow N$. Hence, for every $X \in H$, $P^\star(X)$ also belongs to N, because N is finite and therefore $\sup\{\mu(x) \mid \mu \in P\} = \max\{\mu(x) \mid \mu \in P\}$.

We want to show that there are finitary axiomatizations of these logics and to prove that they are sound and complete with respect to the considered classes of models.

For $s \in [0,1)$, let $s^+ = \min\{r \in N \mid s < r\}$, and if $s \in (0,1]$, let $s^- = \max\{r \in N \mid s > r\}$.
The axiomatization of the logic $LUPP^{Fr(n)}$ includes all the axioms from Section 4, plus one more axiom:

(8) $U_{>s}\alpha \to U_{\geq s^+}\alpha$.

The inference rules of the axiomatization are rules (1) and (2) from Section 4. Consequently, our axiomatization is finite, and the proofs are finite sequences of formulas.

Lemma 6 *(a)* $\vdash U_{>s}\alpha \leftrightarrow U_{\geq s^+}\alpha$,

(b) $\vdash U_{<s}\alpha \leftrightarrow U_{\leq s^-}\alpha$,

(c) $\vdash \bigvee_{s\in N} U_{=s}\alpha$,

(d) $\vdash \underline{\bigvee}_{s\in N} U_{=s}\alpha$.

Proof. Proofs for (a) and (b) are trivial (direct consequences of Axiom 8 including contrapositive).
(c) Clearly $\vdash (U_{\geq 1}\alpha \vee \neg U_{\geq 1}\alpha) \wedge \neg U_{>1}\alpha$. Therefore

$$\vdash (U_{\geq 1}\alpha \wedge \neg U_{>1}\alpha) \vee (\neg U_{\geq 1}\alpha \wedge \neg U_{>1}\alpha).$$

Since $U_{\geq 1}\alpha \wedge \neg U_{>1}\alpha = U_{=1}\alpha$ and $\vdash U_{<1}\alpha \to U_{\leq 1}\alpha$ we have $\vdash U_{=1}\alpha \vee U_{<1}\alpha$. Furthermore, from $\vdash U_{<1}\alpha \leftrightarrow ((U_{\geq 1^-}\alpha \vee \neg U_{\geq 1^-}\alpha) \wedge U_{<1}\alpha)$ and $\vdash (U_{\geq s}\alpha \to U_{\geq s^-}\alpha) \leftrightarrow (U_{<s^-}\alpha \to U_{<s}\alpha)$ we obtain that $\vdash U_{<1}\alpha \leftrightarrow ((U_{\geq 1^-}\alpha \wedge \neg U_{>1^-}\alpha) \vee (U_{<1^-}\alpha \wedge U_{<1}\alpha))$, and $\vdash U_{=1}\alpha \vee U_{=1^-}\alpha \vee U_{<1^-}\alpha$. Finally, we have that $\vdash (\bigvee_{s\in N} U_{=s}\alpha) \vee U_{<0}\alpha$, so $\vdash (\bigvee_{s\in N} U_{=s}\alpha)$.
(d) $U_{=r}\alpha = U_{\geq r}\alpha \wedge \neg U_{>r}\alpha$, so $\vdash U_{=r}\alpha \to \neg U_{=s}\alpha$, for every $s > r$. Similarly, we can prove that $\vdash U_{=r}\alpha \to \neg U_{=s}\alpha$, for every $s < r$. As a consequence, we obtain $\vdash \underline{\bigvee}_{s\in N} U_{=s}\alpha$. □

The proof of the strong completeness theorem is very similar to one presented in Section 5. We will only explain the idea of the proof without going into the details. First, we can prove the soundness theorem and the deduction theorem in a straightforward way. After that, while proving that every consistent set can be extended to a maximal consistent set, we skip the steps where we use infinitary inference rules, i.e. steps 2(b) and 2(c). One more fact needs some explanation. In the proof of the strong completeness theorem we use that if $\sup\{r \mid U_{\geq r}\alpha \in T^\star\} = s$, and $s \in S$, then $U_{\geq s}\alpha \in T^\star$. Now, we have that s must be in a set N, because if $s \notin N$ then there is some $r < s$ such that $r^+ = s^+$, so $T^\star \vdash U_{\geq s^+}\alpha$, but $s < s^+$. Contradiction.

Furthermore, $U_{\geq s}\alpha \in T^\star$ because of Lemma 6(d). The rest of the proof of strong completeness theorem is identical as in Section 5.

7 Conclusion

In this paper, we introduced the logic $LUPP$, whose language is obtained by adding the operators for upper and lower probabilities to propositional logic. We proposed an axiomatization for the logic and proved strong completeness. Since the logic is not compact, the axiomatization contains infinitary rules of inference. Then we simplified the semantics and we achieved compactness using finite sets of probability values for logics $LUPP^{Fr(n)}$. For those logics we provide finitary axiomatizations.

As a topic for further research, we propose developing a first order extension of the logics $LUPP$ and $LUPP^{Fr(n)}$. To the best of our knowledge, there is no axiomatization for first order logics for reasoning about lower and upper probabilities. Note that such a logic would extend classical first order logic, so the set of all valid formulas is not recursively enumerable [1] and no complete finitary axiomatization is possible in this undecidable framework. On the other hand, our completion techniques are already applied to some first order probabilistic logics [15, 23, 26].

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