Comonotone Lower Probabilities for Bivariate and Discrete Structures

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Abstract

Two random variables are called comonotone when there is an increasing relation between them, in the sense that when one of them increases (decreases), the other one also increases (decreases). This notion has been widely investigated in probability theory, and is related to the theory of copulas. This contribution studies the notion of comonotonicity in an imprecise setting. We define comonotone lower probabilities and investigate its characterizations. Also, we provide some sufficient conditions allowing to define a comonotone belief function with fixed marginals and characterize comonotone bivariate p-boxes.

Keywords. Comonotonicity, copulas, lower probabilities, belief functions, p-boxes.

1 Introduction

Random variables are usual tools in probability theory when modeling uncertainty. When dealing with two random variables, Sklar's Theorem [10] tells us that the joint distribution function can be expressed in terms of the marginals by means of a function called copula [7]. Thus, the copula gathers the information concerning the possible dependence between the random variables. When there is no dependence between them, we talk about independent random variables, and the copula associated with those variables is the product. The extreme cases of dependence between variables are related to situations in which either there is an increasing or decreasing relation between them. In the former case, this means that when the value of one variable increases the other variable also increases, while in the second scenario when the value of one variable increases, the value of the other variable decreases. They are referred as comonotone and countermonotone random variables, respectively, and the associated copulas are the minimum and the Łukasiewicz operator.

In this work we shall assume the existence of an imprecisely known probability and we shall use coherent lower probabilities to model it. Lower probabilities are one of the models within the theory of Imprecise Probabilities introduced by Walley [12], as well as belief functions [9], possibilities [3, 13] or uni- and bivariate p-boxes [11, 8], all of them particular families of coherent lower probabilities.

The aim of this paper is to extend the notion of comonotonicity to coherent lower probabilities and to investigate the particular cases in which the lower probability is a belief function or is associated with a bivariate p-box.

After introducing some preliminary notions related to lower probabilities and copulas in Section 2, Section 3 investigates the definition and characterizations of comonotonicity for coherent lower probabilities. We shall see that, in contrast with the precise framework, not any two marginal coherent lower probabilities allow us to define a joint comonotone coherent lower probability. Thus, in Section 4 we investigate some conditions under which this property is satisfied for the particular case of belief functions. In Section 5 we consider bivariate p-boxes and we characterize the conditions they must satisfy to ensure that its associated lower probability is comonotone.

2 Preliminaries

In this section we introduce some preliminary notions that will be useful throughout the paper. First of all we introduce lower probabilities [12], which are very useful to model situations in which a probability is imprecisely defined. Other model related to lower probabilities is that of p-boxes [4], which are used to model the imprecise knowledge to cumulative distribution functions. Univariate p-boxes are connected to belief functions, which play a key role in Shafer's Theory of Evidence [9]. Finally, we also introduce possibility measures [3], which can be embedded both into the Theory of Evidence and the Fuzzy Set Theory.

Secondly we introduce the main notions of the Theory of Copulas [7] and we explain the problem we are dealing in this paper.

2.1 Lower Probabilities

A *lower probability* [12] is a function $\underline{P} : \mathcal{K} \to [0,1]$, where $\mathcal{K} \subseteq \mathcal{P}(\Omega)$. $\underline{P}(A)$ can be interpreted as the subject's supremum acceptable buying price for the bet A, in the sense that we obtain 1 if A happens and 0 otherwise. Any lower probability defines, using a conjugacy relation, an *upper probability* $\overline{P} : \mathcal{K}^c \to [0,1]$, where $\mathcal{K}^c = \{A^c : A \in \mathcal{K}\}$, by:

$$\overline{P}(A) = 1 - \underline{P}(A^c) \quad \forall A \in \mathcal{K}^c.$$

Any lower probability defines a set of probabilities, usually called *credal set*, given by:

$$\mathcal{M}(\underline{P}) = \{P \text{ prob.} \mid \underline{P}(A) \leq P(A) \leq \overline{P}(A)\}.$$

Some consistency requirements are usually imposed on lower probabilities. The most usual one is *coherence*: a lower probability $\underline{P}$ is coherent when

$$\underline{P}(A) = \min_{P \in \mathcal{M}(\underline{P})} P(A) \quad \forall A \subseteq \Omega,$$

It is well-known that any coherent lower probability satisfies $\underline{P}(A) \leq \overline{P}(A)$ whenever $A \in \mathcal{K} \cap \mathcal{K}^c$. Furthermore, any coherent lower probability defined on $\mathcal{K}$ can be extended to a greater domain $\mathcal{K} \subseteq \mathcal{K}'$ by using the natural extension [12]:

$$\underline{E}(A) = \min\{P(A) \mid P \in \mathcal{M}(\underline{P})\}, \ \forall A \in \mathcal{K}'.$$

In this work we consider lower probabilities defined on finite and ordered possibility spaces, denoted by $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}$, called *marginal* lower probabilities, or defined on the cartesian product of two finite and ordered sets, denoted by $\mathcal{X} \times \mathcal{Y} \subseteq \mathbb{R}^2$, called *joint* lower probabilities, where both $\mathcal{X}$ and $\mathcal{Y}$ are finite. In particular, if $\underline{P}_{\text{X,Y}}$ is a joint lower probability defined on $\mathcal{X} \times \mathcal{Y}$, it defines two marginals on $\mathcal{X}$ and $\mathcal{Y}$ by:

$$\underline{P}_{\text{X}}(A) = \underline{P}_{\text{X,Y}}(A \times \mathcal{Y}), \ \forall A \subseteq \mathcal{X}.$$
$$\underline{P}_{\text{Y}}(B) = \underline{P}_{\text{X,Y}}(\mathcal{X} \times B), \ \forall B \subseteq \mathcal{Y}.$$

Uni- and bivariate p-boxes are specific instances of lower probabilities, defined as follows.

Definition 1. *A discrete univariate p-box defined on the ordered*[1] *finite set* $\mathcal{X} = \{x_1, \ldots, x_n\}$ *is a pair of increasing functions* $\underline{F}, \overline{F} : \mathcal{X} \to [0,1]$ *such that* $\underline{F} \leq \overline{F}$ *and* $\underline{F}(x_n) = \overline{F}(x_n) = 1$.

A discrete bivariate p-box defined on the Cartesian product of finite ordered[2] *sets* $\mathcal{X} \times \mathcal{Y} = \{x_1, \ldots, x_n\} \times \{y_1, \ldots, y_m\}$ *is a pair of component-wise increasing functions*[3] $\underline{F}, \overline{F} : \mathcal{X} \times \mathcal{Y} \to [0,1]$ *such that* $\underline{F} \leq \overline{F}$ *and* $\underline{F}(x_n, y_m) = \overline{F}(x_n, y_m) = 1$.

In what remains, and for the sake of simplicity, we avoid the term "discrete" so we will speak about uni- and bivariate p-boxes.

Remark 1. *Note that in the definition of univariate p-box we do not require* $\underline{F}, \overline{F}$ *to satisfy* $\underline{F}(x_1) = \overline{F}(x_1) = 0$. *The reason is that we interpret* $(\underline{F}, \overline{F})$ *as the imprecise observation of a cumulative distribution function* F. *However, cumulative distribution functions* F *defined on a finite space* $\mathcal{X} = \{x_1, \ldots, x_n\}$ *satisfy the properties:* F *is increasing and* $F(x_n) = 1$. *Nevertheless, as soon as* x_1 *has strictly positive probability,* $F(x_1)$ *will be strictly positive. For this reason the property* $\underline{F}(x_1) = \overline{F}(x_1) = 0$ *is not required for univariate p-boxes.*

With a similar reasoning we can justify why $\underline{F}(x_1, y_1) = \overline{F}(x_1, y_1) = 0$ *is not required for bivariate p-boxes* $(\underline{F}, \overline{F})$.

Univariate [11] and bivariate [8] p-boxes can be used to model the imprecise information about (univariate or bivariate) cumulative distribution functions.

Definition 2. *For any* $x \in \mathcal{X} = \{x_1, \ldots, x_n\}$ *and* $y \in \mathcal{Y} = \{y_1, \ldots, y_m\}$, *consider the following notation:*

$$A_x = [x_1, x], \text{ and } A_{x,y} = A_x \times A_y.$$

A univariate p-box defines a coherent lower probability on the domain $\mathcal{K}_1 = \{A_x, A_x^c : x \in \mathcal{X}\}$ *by:*

$$\underline{P}(A_x) = \underline{F}(x) \text{ and } \underline{P}(A_x^c) = 1 - \overline{F}(x).$$

A bivariate p-box defines a lower probability on the domain $\mathcal{K}_2 = \{A_{x,y}, A_{x,y}^c : (x,y) \in \mathcal{X} \times \mathcal{Y}\}$ *by:*

$$\underline{P}(A_{x,y}) = \underline{F}(x,y) \text{ and } \underline{P}(A_{x,y}^c) = 1 - \overline{F}(x,y). \quad (1)$$

Belief functions are another particular case of lower probabilities.

Definition 3. *A lower probability* $\underline{P}$ *on* $\mathcal{P}(\Omega)$ *is called* n-monotone *if and only if:*

$$\underline{P}\left(\cup_{i=1}^p A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,p\}} (-1)^{|I|+1} \underline{P}\left(\cap_{i \in I} A_i\right)$$

for any $2 \leq p \leq n$ *and any* $A_1, \ldots, A_p \subseteq \Omega$. *A lower probability that is* n*-monotone for any* n *is called*

[1] We assume the elements in $\mathcal{X}$ are indexed according to this order, that is, $x_1 < \ldots < x_n$.

[2] Again, we assume the elements in $\mathcal{X}$ and $\mathcal{Y}$ are indexed according to this order: $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_m$.

[3] A function $F : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is component-wise increasing when $F(x, y_i) \leq F(x, y_j)$ for any $x \in \mathcal{X}$ and $i,j \in \{1, \ldots, m\}$ such that $i < j$ and $F(x_i, y) \leq F(x_j, y)$ for any $y \in \mathcal{Y}$ and $i,j \in \{1, \ldots, n\}$ such that $i < j$.

completely monotone or belief function*, and its upper probability is called* plausibility*. Belief and plausibility functions are usually denoted by* Bel *and* Pl*, and they are coherent lower and upper probabilities.*

Using the so-called Möbius inverse, they define a *mass distribution* [3] in the following way:

$$m(A) = \sum_{E \subseteq A} (-1)^{|A \setminus E|} \mathrm{Bel(E)} \quad \forall \mathrm{A} \subseteq \Omega. \qquad (2)$$

A mass distribution $m : \mathcal{P}(\Omega) \to [0,1]$ satisfies $m(\emptyset) = 0$ and $\sum_{E \subseteq \Omega} m(E) = 1$. Conversely, any mass function defines a belief and plausibility functions

$$\mathrm{Bel(A)} = \textstyle\sum_{\mathrm{E} \subseteq \mathrm{A}} \mathrm{m(E)} \quad \forall \mathrm{A} \subseteq \Omega,$$
$$\mathrm{Pl(A)} = \textstyle\sum_{\mathrm{E}:\mathrm{E} \cap \mathrm{A} \neq \emptyset} \mathrm{m(E)} \quad \forall \mathrm{A} \subseteq \Omega.$$

The positivity of the mass m is characteristic of belief functions, in the sense that Eq. (2) is positive if and only if it is applied to a completely monotone lower probability.

Definition 4. *[9] Given a belief function* Bel *with mass distribution m, the elements $E \subseteq \Omega$ with positive mass, $m(E) > 0$, are called* focal elements*, and we will denote by $\mathcal{F}$ the set of focal elements. The union of all the focal sets is called the* core *of* Bel*, and it is denoted by* Core(Bel).

As for lower probabilities, we shall also use the terminology of *marginal* and *joint* to refer to belief functions defined on $\mathcal{X}, \mathcal{Y}$ and $\mathcal{X} \times \mathcal{Y}$, respectively. Any joint belief function Bel defined on $\mathcal{X} \times \mathcal{Y}$ with mass distribution m defines two *marginal belief functions* $\mathrm{Bel}_{\mathrm{X}}$ and $\mathrm{Bel}_{\mathrm{Y}}$ on $\mathcal{X}$ and $\mathcal{Y}$, respectively, with associated mass distributions m_{X} and m_{Y}:

$$m_{\mathrm{X}}(A) = \sum_{E:E^{\downarrow \mathcal{X}} = A} m(E) \text{ and } m_{\mathrm{Y}}(B) = \sum_{E:E^{\downarrow \mathcal{Y}} = B} m(E)$$

for any $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$, and where $E^{\downarrow \mathcal{X}}$ and $E^{\downarrow \mathcal{X}}$ denote the projection of E on spaces $\mathcal{X}$ and $\mathcal{Y}$. Two important models to which we will devote particular attention and that induce belief functions are univariate p-boxes and possibility measures.

From now on, given E with a finite number of elements we will use the following notation:

$$\underline{e} = \min E, \quad \overline{e} = \max E. \qquad (3)$$

Kriegler and Held [5] showed that the lower probability induced by a p-box in the following way:

$$\underline{P}(A) = \inf\{P(A) : \underline{F} \leq F_P \leq \overline{F}\}, \qquad (4)$$

where F_P is the cumulative distribution function associated with P, is indeed a belief function. Such belief function can be computed as Figure 1 shows. Thus, from now on we shall use the term *focal elements of a p-box* to refer to the focal elements of the belief function associated with a p-box using Eq. (4). According to [5], the focal elements of a p-box, named $E_1, \ldots, E_n$, can be ordered such that $\underline{e}_i \leq \underline{e}_{i+1}$ and $\overline{e}_i \leq \overline{e}_{i+1}$. When dealing with focal sets of p-boxes, we will consider that they are indexed according to this ordering. Furthermore, any joint belief function Bel defines a bivariate p-box in the following way:

$$\underline{F}(x,y) = \inf\{F_P(x,y) : P \in \mathcal{M}(\mathrm{Bel})\} = \mathrm{Bel}(\mathrm{A}_{\mathrm{x,y}});$$
$$\overline{F}(x,y) = \sup\{F_P(x,y) : P \in \mathcal{M}(\mathrm{Bel}_{\mathrm{Y}})\} = \mathrm{Pl}(\mathrm{A}_{\mathrm{x,y}});$$

whereas any marginal belief function Bel defines an univariate p-box:

$$\underline{F}_{\mathrm{X}}(x) = \inf\{F_P(x) : P \in \mathcal{M}(\mathrm{Bel}_{\mathrm{X}})\} = \mathrm{Bel}_{\mathrm{X}}(\mathrm{A}_{\mathrm{x}});$$
$$\overline{F}_{\mathrm{X}}(x) = \sup\{F_P(x) : P \in \mathcal{M}(\mathrm{Bel}_{\mathrm{X}})\} = \mathrm{Pl}_{\mathrm{X}}(\mathrm{A}_{\mathrm{x}}).$$

A possibility measure constitutes another important specific case of plausibility function.

Definition 5. *A possibility measure $\Pi : \mathcal{P}(\Omega) \to [0,1]$ is a supremum-preserving map: $\Pi(\cup_{i \in I} A_i) = \sup_{i \in I} \Pi(A_i)$ for any I, $A_i \subseteq \Omega$.*

The conjugate of a possibility, $N(A) = 1 - \Pi(A^c)$ $\forall A \subseteq \Omega$, is a belief function. Its focal elements are nested: if E_1 and E_2 are focal elements, then either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$. Since we are dealing with finite referentials, there are only a finite number of focal sets $E_1, \ldots, E_n$, and for possibility measures we can assume they are indexed such that $E_1 \subseteq \ldots \subseteq E_n$.

2.2 Sklar's Theorem

Sklar's Theorem is an important tool in probability theory that allows a joint cumulative distribution function (cdf for short) to be expressed in terms of the marginals by means of a function called copula.

Definition 6. *[7] A* copula *is a commutative binary operator $C : [0,1]^2 \to [0,1]$ satisfying:*

1. *$C(x,0) = 0$, $C(x,1) = x$ $\forall x \in [0,1]$.*
2. *$C(x_1,y_1) + C(x_2,y_2) \geq C(x_2,y_1) + C(x_1,y_2)$ $\forall x_1,x_2,y_1,y_2 \in [0,1]$ such that $x_1 \leq x_2$, $y_1 \leq y_2$.*

Some classical copulas are the product copula, $\Pi(x,y) = x \cdot y$, the minimum, $M(x,y) = \min(x,y)$, and the Łukasiewicz operator $W(x,y) = \max(x+y-1,0)$. The minimum and Łukasiewicz operators are also called the Fréchet-Hoeffding bounds because any copula satisfies the so-called Fréchet-Hoeffding inequality $M(x,y) \leq C(x,y) \leq W(x,y)$. Copulas play an important roll in the famous Sklar's Theorem.

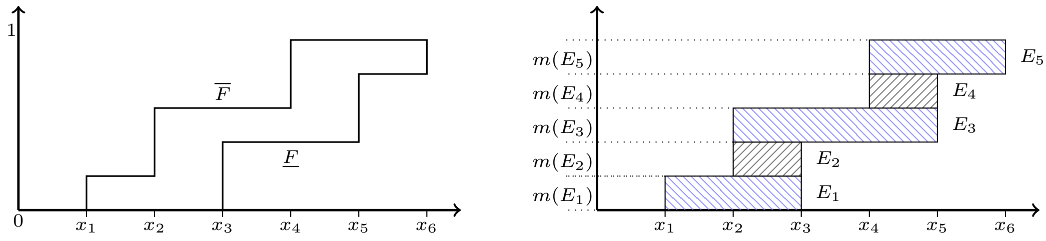

Figure 1: P-box (left) and its associated belief function (right), with focal elements $E_1 = \{x_1, x_2, x_3\}$, $E_2 = \{x_2, x_3\}$, $E_3 = \{x_2, x_3, x_4, x_5\}$, $E_4 = \{x_4, x_5\}$ and $E_5 = \{x_4, x_5, x_6\}$.

Theorem 1. *[10][Sklar's Theorem] Let $F_{X,Y}$ be a joint cdf with marginals F_X and F_Y. Then, there exists a copula C such that*

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)) \ \forall (x, y) \in [0, 1]^2. \quad (5)$$

Conversely, given two marginal cdfs F_X and F_Y and a copula C, they define a joint cdf $F_{X,Y}$ using Eq. (5).

Possibly the most usual application of Sklar's Theorem concerns independent random variables. Two variables X and Y are independent if $F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$, that is, when the copula linking the marginals is the product. Also very important are the cases in which the random variables are coupled by the Fréchet-Hoeffding bounds. Random variables coupled by the minimum (resp., Łukasiewicz operator) are called *comonotone* (resp., *countermonotone*). Comonotone random variables can be characterized in many different ways. For this aim, we first introduce the following notion.

Definition 7. *A subset S of $\mathbb{R}^2$ is* increasing *when for any $(x, y), (u, v) \in S$, $x < u$ implies $y \leq v$, and $y < v$ implies $x \leq u$.*

Then, a pair of random variables (X, Y) is comonotone if it satisfies one, and therefore all, of the following equivalent conditions:

- The copula that links the marginals is the minimum: $F_{X,Y}(x, y) = \min(F_X(x), F_Y(y)) \ \forall (x, y)$.
- The support of (X, Y) is an increasing set on $\mathbb{R}^2$.
- $\forall (x, y) \in \mathbb{R}^2$, either $P(X \leq x, Y > y) = 0$ or $P(X > x, Y \leq y) = 0$.

Remark 2. *The notion of comonotonicity can also be defined for discrete probabilities $P_{X,Y}$, just by substituting the support of (X, Y) by the support of $P_{X,Y}$.*

For example, consider a finite Ω where all its elements have positive probability. Consider the random variables X, Y defined by:

	$\omega \in \Omega_1 \subset \Omega$	$\omega \in \Omega_1^c \subset \Omega$
X	1	2
Y	0	3

Then, the support of (X, Y) is given by $\{(1, 0), (2, 3)\}$, which is an increasing subset of $\mathbb{R}^2$ and therefore (X, Y) is comonotone. In this case, we can also consider the support of $P_{(X,Y)}$, which are the elements (x, y) with positive possibility. In this case, the support of $P_{(X,Y)}$ coincides with the support of (X, Y) and therefore $P_{(X,Y)}$ is comonotone.

When we have imprecise information about the joint or the marginal cdfs or about the copula, Sklar's Theorem cannot be applied. The next Theorem adapts Sklar's Theorem to the imprecise setting, using p-boxes, both uni- and bivariate, and sets of copulas.

Theorem 2. *[6][Imprecise Sklar's Theorem]*

1. *Given two univariate p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ and a set of copulas $\mathcal{C}$, consider:*

   $$\underline{F}(x, y) = \inf_{C \in \mathcal{C}} C(\underline{F}_X(x), \underline{F}_Y(y)) \text{ and } \overline{F}(x, y) = \sup_{C \in \mathcal{C}} C(\overline{F}_X(x), \overline{F}_Y(y)).$$

   Then, they define a bivariate p-box $(\underline{F}, \overline{F})$ whose associated lower probability is coherent.

2. *Given a bivariate p-box $(\underline{F}, \overline{F})$, it could not be possible to express it in terms of the univariate p-boxes and a set of copulas, even when the lower probability associated with $(\underline{F}, \overline{F})$ is coherent.*

In the framework of imprecise probabilities, the notion of independence has been widely investigated [1, 2]. However, those satisfying the factorizing property have the same associated bivariate p-box, and it is obtained by applying the product copula to the marginals p-boxes [6, Prop. 6]:

$$\underline{F}(x, y) = \underline{F}_X(x) \cdot \underline{F}_Y(y) \text{ and } \overline{F}(x, y) = \overline{F}_X(x) \cdot \overline{F}_Y(y)$$

for any x, y. The question now is: what is the meaning of comonotonicity in the imprecise probability setting? As far as we know, this remains unexplored. Thus, the aim of this paper is to define the notion of comonotonicity when dealing with coherent lower probabilities.

3 Comonotone Lower Probabilities

We have seen that comonotonicity in the precise framework can be expressed in three equivalent ways. Now, we shall try to investigate to what extent these conditions, or similar ones, also hold in the case of coherent lower probabilities.

In our framework, we consider a coherent lower probability $\underline{P}$ defined on the power set of $\mathcal{X} \times \mathcal{Y}$. We assume that $\underline{P}$ models the imprecise information about a joint probability $P_{\mathrm{X,Y}}$. The question is: how can we model the additional information that $P_{\mathrm{X,Y}}$ is comonotone?

Definition 8. *A lower probability $\underline{P}$ defined on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is called* comonotone *when any $P \in \mathcal{M}(\underline{P})$ is comonotone.*

This is a straightforward definition, since if $\underline{P}$ models the imprecise information about a comonotone probability $P_{\mathrm{X,Y}}$, all the probabilities compatible with the lower probability should be comonotone.

Example 1. *Consider the lower probability $\underline{P}$ defined on $\{0,1\} \times \{1,2\}$ such that:*

$$\begin{aligned}
&\underline{P}(\{(1,2)\}) = \alpha \in (0, 0'5),\ \underline{P}(\{(0,1)\}) = \beta \in (0, 0'5)\\
&\underline{P}(\{(0,1),(0,2),(1,2)\}) = 1,\\
&\underline{P}(\{(0,1),(0,2),(1,2),(1,1)\}) = 1,\\
&\underline{P}(A) = 0 \text{ otherwise.}
\end{aligned}$$

This lower probability is coherent and its credal set is formed by all the convex combinations of the following precise probabilities:

	$\{(0,1)\}$	$\{(0,2)\}$	$\{(1,2)\}$
P_1	β	$1-\alpha-\beta$	α
P_2	β	0	$1-\beta$
P_3	$1-\alpha$	0	α

Then, the support of any $P \in \mathcal{M}(\underline{P})$ is included in $\{(0,1),(0,2),(1,2)\}$, that is an increasing set, and therefore all the probabilities in $\mathcal{M}(\underline{P})$ are comonotone, and then also is $\underline{P}$.

We now investigate how comonotone coherent lower probabilities can be equivalently expressed. We first express it by means of sets $\{X > x, Y \leq y\}$ and $\{X \leq x, Y > y\}$.

Theorem 3. *A coherent lower probability $\underline{P}$ defined on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is comonotone if and only if any $\forall (x,y) \in \mathcal{X} \times \mathcal{Y}$ either*

$$\begin{aligned}
&\overline{P}(\{(u,v) : u \leq x, v > y\}) = 0 \text{ or}\\
&\overline{P}(\{(u,v) : u > x, v \leq y\}) = 0.
\end{aligned}$$

This theorem shows that the characterization of comonotone random variables in terms of event probabilities also holds in the imprecise case. Now, we are going to see that, if we define the support $\mathrm{supp}(\underline{P})$ of a lower probability $\underline{P}$ by:

$$\mathrm{supp}(\underline{P}) = \bigcup_{P \in \mathcal{M}(\underline{P})} \mathrm{supp}(P),$$

its comonotonocity can also be equivalently expressed in terms of the increasingness of $\mathrm{supp}(\underline{P})$.

Theorem 4. *A coherent lower probability $\underline{P}$ defined on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is comonotone if and only if its support $\mathrm{supp}(\underline{P})$ is an increasing set.*

Therefore, this second equivalent expression also holds for lower probabilities. Now, it only remains to check whether or not the comonotonicity of lower probabilities is related to the copula that links the marginals. The next result shows one implication.

Theorem 5. *Let $\underline{P}$ be a coherent comonotone lower probability defined on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$. If $(\underline{F}, \overline{F})$, $(\underline{F}_{\mathrm{X}}, \overline{F}_{\mathrm{X}})$ and $(\underline{F}_{\mathrm{Y}}, \overline{F}_{\mathrm{Y}})$ denote the bivariate and the marginal univariate p-boxes, respectively, then for any (x,y):*

$$\begin{aligned}
&\underline{F}(x,y) = \min(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(y)) \text{ and}\\
&\overline{F}(x,y) = \min(\overline{F}_{\mathrm{X}}(x), \overline{F}_{\mathrm{Y}}(y)).
\end{aligned}$$

The next example shows that, unfortunately, the converse implication does not hold in general.

Example 2. *Consider the joint coherent lower probability $\underline{P}$ defined on $\{1,2\}^2$ by:*

$$\begin{aligned}
&\underline{P}(\{(1,1),(1,2),(2,2)\}) = \alpha > 0,\\
&\underline{P}(\{(1,1),(2,1),(2,2)\}) = 1-\alpha > 0,\\
&\underline{P}(\{(1,1),(1,2),(2,1),(2,2)\}) = 1,\\
&\underline{P}(A) = 0 \text{ otherwise.}
\end{aligned}$$

Then, regardless of α, $\underline{F} = I_{\{(x,y):x,y \geq 2\}}$ and $\overline{F} = I_{\{(x,y):x,y \geq 1\}}$. Furthermore:

$$\begin{aligned}
&\underline{F}_{\mathrm{X}}(x) = \underline{F}_{\mathrm{Y}}(x) = I_{\{x \geq 2\}}(x) \text{ and}\\
&\overline{F}_{\mathrm{X}}(x) = \overline{F}_{\mathrm{Y}}(x) = I_{\{x \geq 1\}}(x).
\end{aligned}$$

Then:

$$\begin{aligned}
&\underline{F}(x,y) = \min(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(y)) \text{ and}\\
&\overline{F}(x,y) = \min(\overline{F}_{\mathrm{X}}(x), \overline{F}_{\mathrm{Y}}(y)).
\end{aligned}$$

However, $\underline{P}$ is not comonotone because the support of $\underline{P}$ contains the elements $(1,2)$ and $(2,1)$, and this contradicts Theorem 4.

Thus, going from a precise to an imprecise setting, comonotonicity can only be characterized by two equivalent ways: by means of the increasingness of the support or by means of the upper probability of the adequate sets. Indeed, the bivariate p-box of a comonotone lower probability is the minimum of the marginals, but the minimum of two marginal p-boxes will not necessarily generate a comonotone lower probabilities. Figure 2 summarizes the conditions we have seen along this section.

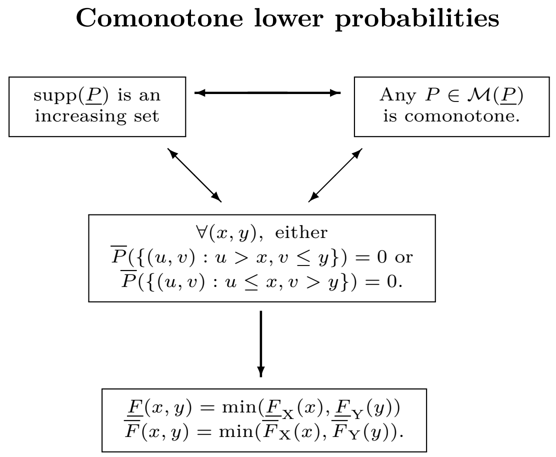

Figure 2: Summary of the conditions for joint comonotone lower probabilities.

4 Comonotone Belief Functions

We now focus on the comonotonicity of belief functions. In this case we have to note that supp(Bel) coincides with Core(Bel), and therefore Theorem 4 can be directly adapted.

Corollary 1. *A belief function* Bel *defined on* $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ *is comonotone if and only if* Core(Bel) *is an increasing set.*

We may think that the converse of Theorem 5 could hold when dealing with belief functions. However, this is not the case, since the lower probability given in Example 2 is in fact a belief function with focal elements $E_1 = \{(1,1),(1,2),(2,2)\}$ and $E_2 = \{(1,1),(2,1),(2,2)\}$ with $m(E_1) = \alpha$ and $m(E_2) = 1-\alpha$, respectively.

Although Section 3 characterized comonotone lower probabilities, did not explore important questions: when and how can we build a comonotone lower probability $\underline{P}$ from marginals $\underline{P}_{\mathrm{X}}$, $\underline{P}_{\mathrm{X}}$? These are the questions we address in this section, for the specific case of belief functions.

Note that those questions can always be answered positively in the precise framework, as it is always possible to define a joint comonotone probability from two marginal probabilities P_{X} and P_{Y}, by simply defining $F_{\mathrm{X,Y}}$ as the minimum of the marginals and then considering the associated probability. Unfortunately, not every marginal lower probabilities allow us to define a comonotone lower probability with the given marginals, even when the lower probabilities are belief functions, as the next example shows.

Example 3. *Let* $\mathrm{Bel}_{\mathrm{X}}$ *and* $\mathrm{Bel}_{\mathrm{Y}}$ *be the marginal belief functions, defined over* $\mathcal{X} = \{1,2,3\}$ *and* $\mathcal{Y} = \{1,2\}$ *with mass distributions*

$$m_{\mathrm{X}}(\{1,2\}) = 0.7,\ m_{\mathrm{X}}(\{1,2,3\}) = 0.3;$$
$$m_{\mathrm{Y}}(\{1\}) = 0.3,\ m_{\mathrm{Y}}(\{2\}) = 0.7.$$

Let us assume that there is a comonotone joint belief function Bel *whose marginals are the belief functions* $\mathrm{Bel}_{\mathrm{X}}, \mathrm{Bel}_{\mathrm{Y}}$ *induced by* m_{X}, m_{Y}. *If this is the case, using Theorem 5, the bivariate p-box induced by* Bel *is the minimum of the marginals* $\underline{F}_X, \overline{F}_X$ *and* $\underline{F}_Y, \overline{F}_Y$. *Then:*

$$\overline{F}(1,2) = \min(\overline{F}_{\mathrm{X}}(1), \overline{F}_{\mathrm{Y}}(2)) = 1.$$

This implies that any focal set E *of* Bel *satisfies* $E \cap \{(1,1),(1,2)\} \neq \emptyset$ *because* $\overline{F}(1,2) = Pl(\{(1,1),(1,2)\})$. *Furthermore,* $(1,2) \in \mathrm{Core(Bel)}$, *because:*

$$Pl(\{(1,1)\}) = \overline{F}(1,1) = \min(\overline{F}_{\mathrm{X}}(1), \overline{F}_{\mathrm{Y}}(1)) = 0.3 < 1 = \overline{F}(1,2),$$

which means that there is a focal element E *such that* $(1,2) \in E$ *and* $(1,1) \notin E$. *Now, since* Bel *is comonotone,* Core(Bel) *is increasing by Corollary 1, and then* $(2,1) \notin \mathrm{Core(Bel)}$, *hence there is no focal element* E *such that* $(2,1) \in E$. *Yet, we have*

$$\underline{F}(2,1) = \mathrm{Bel}(\{(1,1),(2,1)\}) = 0.3,$$

which implies that there is a focal set E *such that* $E \subseteq \{(1,1),(2,1)\}$. *Since* $(2,1) \notin E$, $E = \{(1,1)\}$, *what implies that* $\mathrm{Bel}(\{(1,1)\}) > 0$. *However, if* Bel *is comonotone, it follows that*

$$\mathrm{Bel}(\{(1,1)\}) = \underline{F}(1,1) = \min(\underline{F}_{\mathrm{X}}(1), \underline{F}_{\mathrm{Y}}(1)) = 0,$$

a contradiction showing that there are no comonotone belief functions with marginals $\mathrm{Bel}_{\mathrm{X}}, \mathrm{Bel}_{\mathrm{Y}}$.

This shows that our problem is trickier to answer in the imprecise setting. Below we provide some situations under which a joint comonotone belief function exists with given marginals.

The first case we investigate is when the marginals are possibility measures. Before introducing the main result, note that for any two possibilities having $A_1, \ldots, A_m$ and $B_1, \ldots, B_\ell$ as focal elements, we can always duplicate those elements to build an equivalent mass function (in terms of induced belief function) with focal elements $C_1, \ldots, C_n$ and $D_1, \ldots, D_n$ such that

- $C_i \subseteq C_{i+1}$ and $D_i \subseteq D_{i+1}$ for any $i = 1, \ldots, n-1$.
- $C_i \in \{A_1, \ldots, A_m\}$ and $D_i \in \{B_1, \ldots, B_l\}$ for any $i \in \{1, \ldots, n\}$.

- $m_{\mathrm{X}}(C_i) = m_{\mathrm{Y}}(D_i)$.

The next example illustrates this procedure.

Example 4. *Consider the possibility measures with the following focal sets:*

$$A_1 = \{2\}, \quad A_2 = \{1,2\}, \quad A_3 = \{1,2,3\},$$
$$B_1 = \{1,2\}, \quad B_2 = \{1,2,3,4\},$$

with the following masses:

$$m_{\mathrm{X}}(A_1) = 0.3, \quad m_{\mathrm{X}}(A_2) = 0.5, \quad m_{\mathrm{X}}(A_3) = 0.2.$$
$$m_{\mathrm{Y}}(B_1) = 0.5, \quad m_{\mathrm{Y}}(B_2) = 0.5.$$

Now, we rewrite the focal sets in the following way

	A_1	A_2	A_2	A_3
	B_1	B_1	B_2	B_2
	C_1	C_2	C_3	C_4
	D_1	D_2	D_3	D_4
m	0.3	0.2	0.3	0.2.

We can therefore assume, without loss of generality, that any two possibilities have the same number of focal sets and that their masses coincide.

Proposition 1. *Given two marginal possibility measures, there exists a joint comonotone possibility whose marginals are the original possibility measures.*

This result gives a constructive method for building the joint comonotone possibility. If $A_1 \subseteq \ldots \subseteq A_n$ and $B_1 \subseteq \ldots \subseteq B_n$ denotes the focal elements of m_{X} and m_{Y} such that $m_{\mathrm{X}}(A_i) = m_{\mathrm{Y}}(B_i)$ for $i = 1, \ldots, n$. Using the notation of Eq. (3), Algorithm 1 shows how to define the focal elements and mass function associated with the joint comonotone possibility.

The next example shows how to apply this procedure.

Example 5. *Consider the possibility measures of Example 4. We define the following focal sets for the joint comonotone possibility:*

$$E_1 = \{(2,1),(2,2)\}, \quad E_2 = \{(1,1),(2,1),(2,2)\},$$
$$E_3 = \{(1,1),(2,1),(2,2),(2,3),(2,4)\},$$
$$E_4 = \{(1,1),(2,1),(2,2),(2,3),(2,4),(3,4)\},$$

and the masses are:

	E_1	E_2	E_3	E_4
m	0.3	0.2	0.3	0.2

Let us now look at the case where the focal elements $A_1, \ldots, A_m$ and $B_1, \ldots, B_\ell$ of the marginal belief functions $\mathrm{Bel}_{\mathrm{X}}$ and $\mathrm{Bel}_{\mathrm{Y}}$ can be ordered such that, following notation of Eq. (3), $\underline{a}_i \leq \underline{a}_{i+1}$, $\overline{a}_i \leq \overline{a}_{i+1}$ and $\underline{b}_j \leq \underline{b}_{j+1}$, $\overline{b}_j \leq \overline{b}_{j+1}$ for any $i = 1, \ldots, m-1$ and $j = 1, \ldots, \ell$ and are intervals, in the sense that any

Algorithm 1 Procedure defining focal elements of the joint comonotone possibility

1: **for** $i = 2, \ldots, n$ **do**

$$\mathcal{I}_i = \{(x, \underline{b}_i) : x \in [\underline{a}_i, \underline{a}_{i-1}]\} \cup \{(\underline{a}_{i-1}, y) : y \in [\underline{b}_i, \underline{b}_{i-1}\}$$
$$\cup \{(c, \overline{b}_{i-1}) : x \in [\overline{a}_{i-1}, \overline{a}_i]\} \cup \{(\overline{a}_i, y) : y \in [\overline{b}_{i-1}, \overline{b}_i]\}$$

2: **end for**

3: Define

$$\mathcal{G}_1 = \{(x, \overline{b}_1) : x \in [\underline{a}_1, \overline{a}_1]\} \cup \{(\overline{a}_1, y) : y \in [\underline{b}_1, \overline{b}_{i-1}]\}$$

4: **for** i=2,. . . ,n **do**

$$\mathcal{G}_i = \mathcal{I}_i \cup \mathcal{G}_{i-1}$$

5: **end for**

6: **for** i=1,. . . ,n **do**

$$F_i = \mathcal{G}_i \cap (A_i \times \mathbb{R}) \cap (\mathbb{R} \times B_i)$$
$$m(F_i) = m_{\mathrm{X}}(A_i) = m_{\mathrm{Y}}(B_i)$$

7: **end for**

A_i, B_j contains all elements in $\mathcal{X}, \mathcal{Y}$ between $\underline{a}_i, \overline{a}_i$ and $\underline{b}_j, \overline{b}_j$, respectively. Similarly to focal elements of possibility distributions, those focal elements can be expressed as $\{C_1, \ldots, C_n\}$ and $\{D_1, \ldots, D_n\}$ simply by duplicating elements. Then, they satisfy:

- $\underline{c}_i \leq \underline{c}_{i+1}$, $\overline{c}_i \leq \overline{c}_{i+1}$, $\underline{d}_i \leq \underline{d}_{i+1}$ and $\overline{d}_i \leq \overline{d}_{i+1}$ for any $i \in \{1, \ldots, n\}$.
- $C_i \in \{A_1, \ldots, A_m\}$ and $D_i \in \{B_1, \ldots, B_\ell\}$ for any $i \in \{1, \ldots, n\}$.
- $m_{\mathrm{X}}(C_i) = m_{\mathrm{Y}}(D_i)$ for any $i = 1, \ldots, n$.

Example 6. *Consider the belief functions* $\mathrm{Bel}_{\mathrm{X}}$ *and* $\mathrm{Bel}_{\mathrm{Y}}$ *whose focal elements are:*

$$A_1 = \{0,1\}, \quad A_2 = \{1,2\}, \quad A_3 = \{2,3\} \text{ and}$$
$$B_1 = \{0,1\}, \quad B_2 = \{1,2\},$$

whose masses are:

$$m_{\mathrm{X}}(A_1) = 0.4, \quad m_{\mathrm{X}}(A_2) = 0.3, \quad m_{\mathrm{X}}(A_3) = 0.3;$$
$$m_{\mathrm{Y}}(B_1) = 0.6, \quad m_{\mathrm{Y}}(B_2) = 0.4.$$

We rewrite the focal elements in the following way:

	A_1	A_2	A_2	A_3
	B_1	B_1	B_2	B_2
	C_1	C_2	C_3	C_4
	D_1	D_2	D_3	D_4
m	0.4	0.2	0.1	0.3

Then, from now on we will assume that given two marginal belief functions whose focal sets are intervals ordered through the lattice ordering, both belief *functions have the same number of focal sets and their masses coincide.*

Proposition 2. *Consider two marginal belief functions* Bel_X *and* Bel_Y *with mass distributions* m_X*,* m_Y *whose focal elements* $\mathcal{A} = \{A_1, \ldots, A_n\}$*,* $\mathcal{B} = \{B_1, \ldots, B_n\}$ *are such that* A_i *and* B_i *are intervals and* $m_X(A_i) = m_Y(B_i)$ *for any* $i = 1, \ldots, n$*. If* $\mathcal{A}$ *and* $\mathcal{B}$ *satisfy the following constraints:*

I) $\underline{a}_i \leq \underline{a}_{i+1}$ *and* $\overline{a}_i \leq \overline{a}_{i+1}$ *for any* $i = 1, \ldots, n$*.*

II) $\underline{b}_i \leq \underline{b}_{i+1}$ *and* $\overline{b}_i \leq \overline{b}_{i+1}$ *for any* $i = 1, \ldots, n$*.*

III) If $\overline{a}_i < \underline{a}_j$*, then* $\overline{b}_i \leq \underline{b}_j$*.*

IV) If $\overline{b}_i < \underline{b}_j$*, then* $\overline{a}_i \leq \underline{a}_j$

then, there exists a joint comonotone belief function Bel *such that its marginal masses coincide with* m_X *and* m_Y*.*

Using the notation of Eq. (3), Algorithm 2 shows how to build the focal elements and the mass of the joint comonotone belief function.

Algorithm 2 Procedure defining focal elements of the joint comonotone belief function

1: Define

$$\mathcal{G} = \{(\underline{a}_i, \underline{b}_i), (\overline{a}_i, \overline{b}_i) : i = 1, \ldots, n\}$$

2: Name the elements on $\mathcal{G}$ by:

$$\mathcal{G} = \{(c_1, d_1), \ldots, (c_{2n}, d_{2n})\}$$
$$c_i \leq c_{i+1} \text{ and } d_i \leq d_{i+1} \text{ for } i = 1, \ldots, 2n-1$$

3: **for** $i = 1, \ldots, 2n$-1 **do**

$$\mathcal{I}_i = \{(x, d_k) : x \in [c_k, c_{k+1}]\} \cup \{c_{k+1}, y) : y \in [d_k, d_{k+1}\}$$

4: **end for**

5: **for** i=1,...,n **do**

$$E_i = \cup_{(\underline{a}_i, \underline{b}_i) \leq (c_k, d_k) < (\overline{a}_i, \overline{a}_i)} \mathcal{I}_k$$
$$m(E_i) = m_X(A_i) = m_Y(B_i)$$

6: **end for**

The next example shows how this algorithm is applied.

Example 7. *Let us continue Example 6. We build the following focal sets for the joint belief function:*

$$E_1 = \{(0,0),(1,0),(1,1)\},\ E_2 = \{(1,0),(1,1),(2,1)\},$$
$$E_3 = \{(1,1),(2,1),(2,2)\},\ E_4 = \{(2,1),(2,2),(2,3)\}.$$

Now, we assigns the following masses:

	E_1	E_2	E_3	E_4
m	0.4	0.2	0.1	0.3

This joint belief function is comonotone and its marginals coincide with Bel_X *and* Bel_Y*.*

The condition in Proposition 2 that focal sets should be intervals is essential, as the next example shows.

Example 8. *Consider two mass functions* m_X *and* m_Y *with* $\mathcal{A} = \{A_1, A_2\}$ *and* $\mathcal{B} = \{B\}$*, where:*

$$A_1 = \{1, 3\}, \quad A_2 = \{2, 4\}, \quad B = \{1, 1, \ldots, n-1, n\}$$

for $n > 3$*.* $\mathcal{A}$ *and* $\mathcal{B}$ *satisfy all the conditions of Proposition 2, except for being intervals. However, there is no joint comonotone belief functions having those marginals. Indeed, following Algorithm 2, such a joint would have two focal elements* E_1, E_2 *with projections* A_1, B *and* A_2, B*, respectively, and such that* $E_1 \cup E_2$ *is increasing. Now, for any* $x \in \{1, 1, \ldots, n-1, n\}$*,* $E_1 \cup E_2$ *must contain, at least for one* x*, any the following pair:* $(x, 1)$ *and* $(x, 2)$*,* $(x, 1)$ *and* $(x, 4)$*,* $(x, 3)$ *and* $(x, 2)$*, or* $(x, 3)$ *and* $(x, 4)$*, for* E_1, E_2 *to have the required projections. If we take any two of those pairs for two different* $x \leq y$ *in* $\{1, 1, \ldots, n-1, n\}$*, then they form a non-increasing set. For example, take* $(x, 1), (x, 4)$ *and* $(y, 3), (y, 4)$*, we have* $(x, 4) \not\leq (y, 3)$*. Hence it is not possible to build a comonotone joint belief from* m_X *and* m_Y*.*

We have seen conditions under which, given marginal belief functions, it is possible to define a joint comonotone belief function. However, the next example shows that this joint comonotone belief function is not unique.

Example 9. *Consider the marginal belief functions* Bel_X *and* Bel_Y *with mass distributions* m_X *and* m_Y*, given by:*

$$m_X(\{1, 2\}) = m_Y(\{1, 2\}) = 1.$$

In this case, we can define three joint belief functions that are comonotone: if we denote their masses by m*,* m' *and* m''*, they are given by:*

$$m(\{(1,1),(2,2)\}) = m'(\{(1,1),(2,2),(1,2)\}) = m''(\{(1,1),(2,2),(2,1)\}) = 1.$$

5 Comonotone p-boxes

Consider now a bivariate p-box $(\underline{F}, \overline{F})$ defined on $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{Y} = \{y_1, \ldots, y_m\}$. We have already said that bivariate p-boxes define a lower probability $\underline{P}$ on the set $\mathcal{K}_2$ following Eq. (1).

Consider the natural extension of $\underline{P}$ to $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$, and we are going to investigate whether it is comonotone or not. During this section, and for the sake of simplicity, we shall assume that $n, m > 1$, $\overline{P}(\{x_i\}) > 0$ and $\overline{P}(\{y_j\}) > 0$ for any $i = 1, \ldots, n$ and $j = 1, \ldots, m$. This implies that $\overline{F}_{\mathrm{X}}(x_i) > \underline{F}_{\mathrm{X}}(x_{i-1})$ and $\overline{F}_{\mathrm{Y}}(y_j) > \underline{F}_{\mathrm{Y}}(y_{j-1})$ for $i = 2, \ldots, n$ and $j = 2, \ldots, m$.

In [8] it is argued that some notions like "avoiding sure loss", "coherence" or "2-mononicity" of a bivariate p-box is given in terms of its associated lower probability (given in Eq. (1)).

Definition 9. *A coherent bivariate p-box is comonotone when its associated lower probability is comonotone.*

Next results give two characterizations of comonotone bivariate p-boxes. The first one establishes the form of the bivariate p-box.

Proposition 3. *Let $(\underline{F}, \overline{F})$ be a coherent bivariate p-box defined on $\mathcal{X} \times \mathcal{Y}$. Then, it is comonotone if and only if there is an increasing set $S \subseteq \mathcal{X} \times \mathcal{Y}$, named $S = \{(u_1, v_1), \ldots, (u_k, v_k)\}$, such that:*

S.1 *The X and Y projections of S are $\mathcal{X}$ and $\mathcal{Y}$.*

S.2 *If $(x_i, y_j) \in S$ and $(x_{i+1}, y_j) \notin S$, then*

$$\underline{F}(x_i, y_j) = \underline{F}(x_{i+1}, y_j) = \cdots = \underline{F}(x_n, y_j) = \overline{F}(x_i, y_j) = \overline{F}(x_{i+1}, y_j) = \cdots = \overline{F}(x_n, y_j).$$

S.3 *If $(x_i, y_j) \in S$ and $(x_i, y_{j+1}) \notin S$, then*

$$\underline{F}(x_i, y_j) = \underline{F}(x_i, y_{j+1}) = \cdots = \underline{F}(x_i, y_m) = \overline{F}(x_i, y_j) = \overline{F}(x_i, y_{j+1}) = \cdots = \overline{F}(x_i, y_m).$$

The second result characterizes comonotone coherent bivariate p-boxes in terms of the belief functions associated with its marginal p-boxes.

Theorem 6. *Let $(\underline{F}, \overline{F})$ be a coherent bivariate p-box defined on $\mathcal{X} \times \mathcal{Y}$. Denote by $(\underline{F}_{\mathrm{X}}, \overline{F}_{\mathrm{X}})$ and $(\underline{F}_{\mathrm{Y}}, \overline{F}_{\mathrm{Y}})$ its marginal p-boxes, and by $\mathrm{Bel}_{\mathrm{X}}$ and $\mathrm{Bel}_{\mathrm{Y}}$ the belief functions associated with the marginal p-boxes. Then, $(\underline{F}, \overline{F})$ is comonotone if and only if one of the following conditions are satisfied:*

1. $\mathrm{Bel}_{\mathrm{X}}$ *is precise with positive probability in $\{x_1\}, \ldots, \{x_n\}$.* $\mathrm{Bel}_{\mathrm{X}}$ *and* $\mathrm{Bel}_{\mathrm{Y}}$ *satisfy the following conditions:*
   - *The focal elements of* $\mathrm{Bel}_{\mathrm{Y}}$ *are $\{y_1\}, \ldots, \{y_{l-1}\}$, where $l \in \{1, \ldots, m\}$, and $B_1, \ldots, B_s$, where $y_l = \min_{i=1,\ldots,s} \min B_i$ and, $\cup_{i=1}^{s} B_i = \{y_{l+1}, \ldots, y_m\}$.*
   - $m_{\mathrm{X}}(\{x_n\}) \geq \sum_{i=1}^{s} m_{\mathrm{Y}}(B_i) - m_{\mathrm{Y}}(\{y_l\})$.

Focal elements of $\mathrm{Bel}_{\mathrm{X}}$:

Focal elements of $\mathrm{Bel}_{\mathrm{Y}}$:

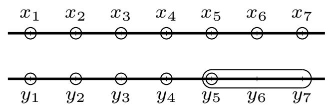

Figure 3: Example of belief functions that allow to build a comonotone bivariate p-box. According to Theorem 6, $m_{\mathrm{Y}}(\{y_5, y_6, y_7\}) \leq m_{\mathrm{X}}(\{x_7\})$ must hold.

2. *Condition 1 holds when we exchange the role of* $\mathrm{Bel}_{\mathrm{X}}$ *and* $\mathrm{Bel}_{\mathrm{Y}}$*.*

Using the previous theorem, we can state the following corollary.

Corollary 2. *If a bivariate p-box is comonotone, its associated lower probability is a belief function.*

From this result we know that any comonotone coherent bivariate p-box can be built with the adequate belief functions. We can also deduce that most bivariate p-boxes will not be comonotone. Thus, under the interpretation of Definitions 8 and 9, bivariate p-boxes do not seem to be adequate to model comonotonicity.

Example 10. *Figure 3 shows an example of marginal belief functions satisfying the conditions of Theorem 6. Assume that the masses are the following:*

	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$	$\{x_6\}$	$\{x_7\}$
m_{X}	0.12	0.15	0.22	0.13	0.1	0.08	0.2

	$\{y_1\}$	$\{y_2\}$	$\{y_3\}$	$\{y_4\}$	$\{y_5\}$	$\{y_5, y_6, y_7\}$
m_{Y}	0.17	0.15	0.15	0.18	0.2	0.15

Then, the comonotone bivariate p-box has the following focal elements:

$$E_1 = \{(x_1, y_1)\}, \quad E_2 = \{(x_2, y_1)\}, \quad E_3 = \{(x_2, y_2)\},$$
$$E_4 = \{(x_3, y_2)\}, \quad E_5 = \{(x_3, y_3)\}, \quad E_6 = \{(x_3, y_4)\},$$
$$E_7 = \{(x_4, y_4)\}, \quad E_8 = \{(x_5, y_4)\}, \quad E_9 = \{(x_5, y_5)\},$$
$$E_{10} = \{(x_6, y_5)\}, \quad E_{11} = \{(x_7, y_5)\},$$
$$E_{12} = \{x_7\} \times \{y_5, y_6, y_7\}.$$

Their masses are:

	E_1	E_2	E_3	E_4	E_5	E_6
m	0.12	0.05	0.1	0.05	0.15	0.02

	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}
m	0.13	0.03	0.07	0.08	0.05	0.2

Note again that the set S of Proposition 3 is the core of Bel*. It can be seen in Figure 4.*

6 Conclusions

This paper investigates the notion of comonotonicity for coherent lower probabilities. We have seen that

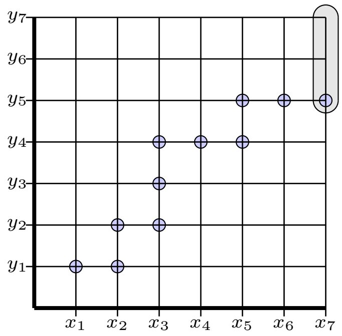

Figure 4: Core of the belief function that defines a comonotone bivariate p-box.

the comonotonicity of a coherent lower probability can be expressed in two equivalent ways: by means of the increasingness of its support or by means of the upper probability of the sets $\{(u, v) : u > x, v \leq y\}$ and $\{(u, v) : u \leq x, v > y\}$. Furthermore, the bivariate p-box associated with a comonotone coherent lower probability can be expressed as the minimum of the marginal p-boxes. However, in contrast to the precise setting, the converse does not hold in general.

Another important difference between precise and imprecise frameworks is that in the former any pair of marginal probabilities admits the definition of a joint comonotone probability with the fixed marginals. This is not the case of lower probabilities, not even when they are belief functions. Nevertheless, such a property does hold for possibility measures and for univariate p-boxes satisfying some additional restrictions.

Unfortunately, we have also seen that bivariate p-boxes, except in very special cases, do not seem to be adequate to model comonotonicity because they impose very strong conditions, like for instance one of the marginals must be precise. Then, in contrast to the precise framework where bivariate distribution functions express the information about comonotonicity, this is not the case of bivariate p-boxes.

One interesting open problem is to investigate the meaning of comonotonicity for a more general framework, that of lower previsions. Although independent products satisfying the factorizing property have the same associated bivariate p-box, in the general framework of lower prevision they are no longer equivalent. It would not be surprisingly that comonotonicity could be extended in many different ways.

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References

[1] I. Couso, S. Moral and P. Walley. A survey of concepts of independence for imprecise probabilities. *Risk Decision and Policy*, 5:165–181, 2000.

[2] G. de Cooman, E. Miranda and M. Zaffalon. Independent natural extension. *Artificial Intelligence*, 175(12-13):1911–1950, 2011.

[3] D. Dubois and H. Prade. *Possibility theory: an approach to computerized processing of uncertainty.* Plenum Press, New York. 1988.

[4] S. Ferson, V. Kreinovich, L. Ginzburg, D. S. Myers and K. Sentz. Constructing probability boxes and Dempster-Shafer structures. Technical Report SAND2002-4015, Sandia National Laboratories, 2003.

[5] E. Kriegler and H. Held. Utilizing random sets for the estimation of future climate change. *International Journal of Approximate Reasoning*, 39:185–209, 2005.

[6] I. Montes, E. Miranda, R. Pelessoni and P. Vicig. Sklar's Theorem in an imprecise setting. *Fuzzy Sets and Systems*, DOI: 10.1016/j.fss.2014,10.007, 2014.

[7] R. Nelsen. *An introduction to copulas.* Springer, New York. 1999.

[8] R. Pelessoni, P. Vicig, I. Montes and E. Miranda. Bivariate p-boxes. *Submitted to publication*, 2014.

[9] G. Shafer. *A mathematical theory of evidence.* Princeton University Press, Princerton, NJ. 1976.

[10] A. Sklar. Fonctions de répartition à n-dimensions et leurs marges. *Publications de l'Institute de Statistique de l'Université de Paris*, 8:229–231, 1959.

[11] M. C. M. Troffaes and S. Destercke. Probability boxes on totally preordered space for multivariate modelling. *International Journal of Approximate Reasoning*, 52(6):767–791, 2011.

[12] P. Walley. *Statistical reasoning with imprecise probabilities.* Chapman and Hall, New York. 1991.

[13] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.