Coherent Conditional Measures of Risk Defined by the Choquet Integral with respect to Hausdorff Outer Measures and Dependent Risks

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Abstract

Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For every set B of $\mathbf{B}$ with positive and finite Hausdorff outer measure in its Hausdorff dimension, a coherent conditional measure of risk is defined as the Choquet integral with respect to Hausdorff outer measure. Two risks are defined to be s-independent if the atoms of the classes generated by their weak upper level sets are s-independent. The given notion permits to capture dependence between risks that are stochastically independent according to the axiomatic definition. Two risks which are surjective and injective are proven to be s-dependent and a sufficient condition is given such that s-independent simple risks satisfy the factorization property of their joint coherent measures of risk.

Keywords. Coherent conditional measures of risk, Hausdorff outer measures, Choquet integral, stochastic dependence.

1 Introduction

Partial knowledge is a natural interpretation of conditional probability. This interpretation can be formalized in a different way in the axiomatic approach (see Billingsley [2]) and in the subjective approach (de Finetti [3], [4], Regazzini [19], Walley [21]), where conditional probability is respectively defined by the Radon-Nikodym derivative or by the axioms of coherence. In both cases conditional probability is obtained as the restriction of conditional expectation or conditional prevision to the class of indicator functions of events. For a comparison between the two different approaches see Doria [6]. In the axiomatic approach conditional expectation is defined with respect to a σ-field $\mathbf{G}$ of conditioning events by the Radon-Nikodym derivative while in the subjective approach proposed by Walley conditional prevision is defined with respect to a partition $\mathbf{B}$; the definitions of conditional expectation and coherent linear conditional prevision can be compared when the σ-field $\mathbf{G}$ is generated by the partition $\mathbf{B}$. In particular, given a probability space $(\Omega, \mathbf{F}, P)$, let $\mathbf{G}$ be equal or contained in the σ-field generated by a countable class S of subsets of $\mathbf{F}$ and let $\mathbf{B}$ be the partition of the atoms generated by the class S. Denote $\Omega' = \mathbf{B}$, $P(A|\mathbf{B})$ the class of all $P(A|B)$ with $B \in \mathbf{B}$ and φ_B the function from Ω to Ω' that associates to every $\omega \in \Omega$ the atom B of the partition $\mathbf{B}$ that contains ω. Then we have that $P(X|\mathbf{G}) = P(X|\mathbf{B}) \circ \varphi_B$ for every random variable $\in L(B)$ [15, p.262].

Let $\mathbf{F}$ and $\mathbf{G}$ be two σ-fields of subsets of Ω with $\mathbf{G}$ contained in $\mathbf{F}$ and let X be an integrable random variable on $(\Omega, \mathbf{F}, P)$. Let P be a probability measure on $\mathbf{F}$; define a measure ν on $\mathbf{G}$ by $\nu(G) = \int_G X dP$. This measure is finite and absolutely continuous with respect to P. So there exists a function, the Radon-Nikodym derivative denoted by $E[X|\mathbf{G}]$, defined on Ω, $\mathbf{G}$-measurable, integrable and satisfying the functional equation

$$\int_G E[X|\mathbf{G}]dP = \int_G XdP \text{ with } G \text{ in } \mathbf{G}.$$

This function is unique up to a set of P-measure zero and it is a version of the conditional expected value.

If X is the indicator function of any event A belonging to $\mathbf{F}$ then $E[X|\mathbf{G}] = E[A|\mathbf{G}] = P[A|\mathbf{G}]$ is a version of the conditional probability.

In Doria [8], [10], [11], [12] it has been proven that conditional expectation, defined by the Radon-Nikodym derivative may fail to be coherent and a new model of coherent conditional previsions, based on Hausdorff outer measures, has been introduced.

In [2, Example 33.11] it is shown that the interpretation of conditional probability in terms of partial knowledge breaks down in certain cases. A probability space $(\Omega, \mathbf{F}, P)$ can be used to represent a random phenomenon or an experiment whose outcome is drawn according to the probability given by P. Partial information about the experiment can be represented by a sub σ-field $\mathbf{G}$ of $\mathbf{F}$ in the following way: an observer does not know which ω has been drawn but he knows for each $H \in \mathbf{G}$, if ω belongs to H or if ω belongs to H^c. A sub σ-field $\mathbf{G}$ of $\mathbf{F}$ can be identified as partial information about the random experiment, and, fixed A in $\mathbf{F}$, conditional probability can be used to represent partial knowledge about A given the information on $\mathbf{G}$.

A concept related to the definition of conditional probability is stochastic independence for events and for random variables based on the factorization property ([2, p.48]). In particular two random variables are stochastically independent, in the axiomatic approach, if the σ-fields generated by them are independent. As a consequence we obtain that for independent random variables the joint distribution is equal to the product of the marginal distributions.

In a probability space (Ω, F, P), if partial information is represented by a sub σ-field G and conditional probability is defined by the Radon-Nykodim derivative, denoted by $P[A|\mathbf{G}]$, by the standard definition [2, p.52] we have that an event A is independent from the σ-field $\mathbf{G}$ if it is independent from each $H \in \mathbf{G}$, that is $P[A|\mathbf{G}] = P(A)$ with probability 1.

If $\mathbf{G} = \{\Omega, \oslash\}$ then $P[A|\mathbf{G}](\omega) = P(A)$ for every $A \in \mathbf{F}$ and for every $\omega \in \Omega$.

Example 1 *Let Ω = [0,1], let $\boldsymbol{F}$ be the Borel σ-field of [0,1] and let P be the Lebesgue measure on $\boldsymbol{F}$. Let $\boldsymbol{G}$ be the sub σ-field of sets that are either countable or co-countable. Then $P(A)$ is a version of the conditional probability $P[A|\boldsymbol{G}]$ defined by the Radon-Nikodym derivative because $P(G)$ is either 0 or 1 for evey $G \in \boldsymbol{G}$. So an event A is independent from the information represented by $\boldsymbol{G}$ and this is a contradiction according to the fact that the information represented by $\boldsymbol{G}$ is complete since $\boldsymbol{G}$ contains all singletons of Ω.*

In the subjective approach the concept of epistemic independence with respect to upper and lower probabilities has proposed by Walley [21]. It is based on the notion of irrelevance; given two events A and B, we say that B is irrelevant to A when $\overline{P}(A|B) = \overline{P}(A|B^c) = \overline{P}(A)$ and $\underline{P}(A|B) = \underline{P}(A|B^c) = \underline{P}(A)$.

The events A and B are epistemically independent when B is irrelevant to A and A is irrelevant to B. As a consequence of this definition we can obtain that the factorization property $P(AB) = P(A)P(B)$, which constitutes the standard definition of independence for events, holds both for $P = \overline{P}$ and $P = \underline{P}$. In a continuous probabilistic space (Ω, F, P), where Ω is equal to $[0,1]^n$ and the probability is usually assumed equal to the Lebesgue measure on Ω, we have that the finite, countable and fractal sets (i.e. the sets with Hausdorff dimension non- integer) have probability equal to zero. For these sets the standard definition of independence, given by the factorization property, is always satisfied since both members of the equality are zero. In Theorem 6 of [9] we prove that an event B is always irrelevant, according to the definition of Walley, to an event A if $dim_H(A) < dim_H(B) < dim_H(\Omega)$ and A and B have positive and finite Hausdorff outer measures in their dimensions; moreover if A and B are disjoint then they are epistemically independent. Thus logical independence is not a necessary condition for epistemic independence.

To avoid these problems the notions of s-irrelevance and s-independence with respect to upper and lower conditional probabilities assigned by a class of Hausdorff outer and inner measures are proposed to test independence ([7], [8], [9]). The definitions of s-independence and s-irrelevance are based on the fact that epistemic independence and irrelevance must be tested for events A and B, such that they and their intersection AB, have the same Hausdorff dimension.

In this paper coherent conditional measures of risk are defined equal to coherent upper conditional previsions defined by Hausdorff measures when the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension. The notion of s-irrelevance and s-independence for risks, represented by bounded random variables, are proposed to capture dependence.

2 Coherent Conditional Measures of Risk Defined by the Choquet Integral with respect to Hausdorff Outer Measures

Let (Ω, d) be a metric space and let $\mathbf{B}$ be a partition of Ω. For every $B \in \mathbf{B}$ with positive and finite Hausdorff outer measure in its Hausdorff dimension s a coherent conditional measure of risk is defined by the Choquet integral with respect to Hausdorff outer measure h^s.

2.1 Coherent Upper Conditional Previsions Defined by the Choquet Integral with respect to Hausdorff Outer Measures

A *risk* or *bounded random variable* is a function $X : \Omega \to \Re$ and $L(\Omega)$ is the class of all bounded random variables defined on Ω. For every $B \in \mathbf{B}$ denote by $X|B$ the restriction of X to B and by $\sup(X|B)$ the supremum value that X assumes on B. Let $L(B)$ be the class of all bounded random variables $X|B$. Denote by I_A the indicator function of any event $A \in \wp(B)$, i.e. $I_A(\omega) = 1$ if $\omega \in A$ and $I_A(\omega) = 0$ if $\omega \in A^c$.

Let $\delta > 0$ and let s be a non-negative number. The *diameter* of a non empty set U of Ω is defined as $|U| = \sup\{d(x,y) : x, y \in U\}$ and if a subset A of Ω is such that $A \subseteq \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, the class $\{U_i\}$ is called a δ-cover of A.

The *Hausdorff s-dimensional outer measure* of A, denoted by $h^s(A)$, is defined on $\wp(\Omega)$, the class of all subsets of Ω, as

$$h^s(A) = \lim_{\delta \to 0} \inf \sum_{i=1}^{+\infty} |U_i|^s .$$

where the infimum is over all δ-covers $\{U_i\}$ of the set A.

For the definition of Hausdorff outer measure and its basic properties see Rogers [20] and Falconer [14].

The *Hausdorff dimension* of a set A, $dim_H(A)$, is defined as the unique value, such that

$$h^s(A) = +\infty \text{ if } 0 \leq s < dim_H(A),$$
$$h^s(A) = 0 \text{ if } dim_H(A) < s < +\infty.$$

If $0 < h^s(A) < +\infty$ then $dim_H(A) = s$ (the converse is not true). In any metric space a finite non-empty subset A of Ω has positive and finite counting measure h^0 so the Hausdorff dimension of a finite set is 0.

We assume that the Hausdorff dimension of the empty-set $\oslash$ is $-\infty$ so that no set has Hausdorff dimension equal to the Hausdorff dimension of the empty-set.

Hausdorff s-dimensional outer measures are submodular, continuous from below and their restrictions to the Borel σ-field which is the σ-field generated by open sets of the *metric topology*, induced by the metric d, are countably additive.

If $B \in$ **B** is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s the monotone set function μ_B^* is defined for every $A \in \wp(B)$ by $\mu_B^*(A) = \frac{h^s(AB)}{h^s(B)}$.

If X is a bounded random variable thus there exists a constant k such that $\widetilde{X} = X + k \geq 0$ and the *decreasing distribution function* of $\widetilde{X}$ with respect to μ_B^* is $G_{\mu_B^*, \widetilde{X}}(x) = G_{\mu_B^*, X}(x-k) = \mu_B^*\{X|B > x - k\}$ for every real number x [5, Proposition 4.1].

The Choquet integral [5] of a bounded random variable X with respect to μ_B^* is defined by

$$\int X d\mu_B^* = \int_0^{+\infty} \mu_B^* \left\{\omega \in B : \widetilde{X}(\omega) \geq x\right\} dx$$
$$= \frac{1}{h^s(B)} \int_0^{+\infty} h^s \{\omega \in B : X(\omega) \geq x\} dx.$$

If B is finite then μ_B^* is the counting measure defined on the field $\wp(B)$. If the atoms A_i are enumerated so that $x_i = X(A_i)$ are in in descending order, i.e. $x_1 \geq x_2 \geq ... \geq x_n$ and $x_{n+1} = 0$ the Choquet integral with respect to μ is given by

$$\int X d\mu = \sum_{i=1}^{n} (x_i - x_{i+1}) \mu_\Omega^*(S_i)$$

where $S_i = A_1 \cup A_2 ... \cup A_i$, and $x_{n+1} = 0$.

In [12] a model of coherent upper conditional prevision based on Hausdorff outer measure has been defined.

Theorem 1 *Let m be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B. Then for each $B \in \boldsymbol{B}$ denote by s the Hausdorff dimension of B and by h^s the Hausdorff s-dimensional outer measure. The functionals $\overline{P}(X|B)$ defined on $L(B)$ by*

$$\overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s \text{ if } 0 < h^s(B) < +\infty$$

and by

$$\overline{P}(X|B) = m(XB) \text{ if } h^s(B) = 0, +\infty$$

are separately coherent upper conditional previsions.

Definition 1 *Given a partition $\boldsymbol{B}$ of Ω and a random variable $X \in L(\Omega)$ a coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ is a random variable on Ω equal to $\overline{P}(X|B)$ if $\omega \in B$. The random variable $\overline{P}(X|\boldsymbol{B})$ is separately coherent if all the $\overline{P}(X|B)$ are separately coherent.*

Definition 2 *Given a partition $\boldsymbol{B}$ of Ω and a random variable $X \in L(\Omega)$, X is $\boldsymbol{B}$-measurable if it is constant on the sets of the partition $\boldsymbol{B}$.*

By Definition 1 the random variable $\overline{P}(X|\mathbf{B})$ is **B**-measurable since it is constant on the atoms of the partition **B**.

2.2 Coherent Conditional Measures of Risk

Coherent measures of risk are introduced in Artzener et al. [1] to manage risks.

Correspondence between the concepts of upper prevision and coherent measure of risk has been underlined by Pelessoni and Vicig [17] and Maaß [16].

In Pelessoni and Vicig [18] risk measures have been interpreted as coherent conditional previsions.

In this section, given a set $B \in \mathbf{B}$ with positive and finite Hausdorff outer measure in its Hausdorff dimension s, a coherent conditional measure of risk $\rho(\cdot|B)$

is defined as the Choquet integral with respect to Hausdorff outer measure h^s.

The advantage to define coherent measures of risk by Hausdorff outer measures is that they can be represented as Choquet integral since Hausdorff outer measures are submodular (see Doria [13, Proposition 1]); moreover coherent measures of risk defined with respect to Hausdorff outer measures are comonotonically additive and continuous from below (see Theorem 2).

Definition 3 *A coherent conditional measure of risk $\rho(\cdot|B)$ is a functional on $L(B)$ such that the following axioms are satisfied for every $X, Y \in L(B)$ and strictly positive λ:*

(i) monotonicity $X \leq Y$ implies $\rho(X|B) \leq \rho(Y|B)$;

(ii) translation invariance $\rho(X+h|B) = \rho(X|B) + h$;

(iii) subadditivity $\rho(X+Y|B) \leq \rho(X|B) + \rho(Y|B)$

(iv) positive homogeneity $\rho(\lambda X|B) = \lambda\rho(X|B)$

Two risks X and $Y \in L(B)$ are *comonotonic* if,

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0 \quad \forall \omega_1, \omega_2 \in B.$$

Definition 4 *A coherent conditional measure of risk $\rho(\cdot|B)$ on $L(B)$ is*

(v) comonotonically additive if and $\rho(X+Y|B) = \rho(X|B) + \rho(Y|B)$ for every comonotonic risks X and Y;

(vi) continuous from below if $lim_{n\to\infty}\rho(X_n|B) = \rho(X|B)$ if X_n is an increasing sequence of risks $\in L(B)$ converging to X.

In [12] it has been proven that if B is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension, the coherent upper conditional prevision defined in Theorem 1 is comonotonically additive and continuous from below. So the following result holds:

Theorem 2 *Let $B \in \boldsymbol{B}$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s, i.e. $0 < h^s(B) < +\infty$; the functional $\rho(\cdot|B)$ defined on $L(B)$ by*

$$\rho(X|B) = \overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s$$

is a coherent conditional measure of risk, which is comonotonically additive and continuous from below.

3 S-Independence for Risks

In Doria [7], [8], [9] the notions of s-irrelevance and s-independence with respect to conditional probabilities assigned by a class of Hausdorff dimensional measures have been introduced.

Definition 5 *Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension and let E and $F \in \wp(\Omega)$. E and F are s-independent if the following conditions hold*

(s1) $dim_H E = dim_H F = dim_H E \cap F$,

(s2) $\overline{P}(E|F) = \overline{P}(E|F^c) = \overline{P}(E)$,

$\underline{P}(E|F) = \underline{P}(E|F^c) = \underline{P}(E)$,

(s3) $\overline{P}(F|E) = \overline{P}(F|E^c) = \overline{P}(F)$,

$\underline{P}(F|E) = \underline{P}(F|E^c) = \underline{P}(F)$,

F is s-irrelevant to E if conditions s1) and s2) hold and E is s-irrelevant to F if the conditions s1) and s3) hold.

We can observe that the notion of s-irrelevance is not symmetric, that is E can be s-irrelevant to F because conditions (s1) and (s2) hold but F is not s-irrelevant to E since condition (s3) does not hold.

In the sequel we assume that Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension.

In [8, Theorem 4] the following result is proven:

Corollary 1 *If E is s-irrelevant to F then $\overline{P}$ satisfies the factorization property, i.e. $\overline{P}(E \cap F) = \overline{P}(E)\overline{P}(F)$*

Theorem 3 *Let $F \in \wp(\Omega)$ then F and Ω are s-independent if and only if $dim_H(F) = dim_H(\Omega)$.*

Proof. If E is equal to Ω conditions $(s1), (s2), (s3)$ become

(s1) $dim_H(\Omega) = dim_H(F)$,

(s2) $\overline{P}(\Omega|F) = \overline{P}(\Omega|F^c) = \overline{P}(\Omega)$,

$\underline{P}(\Omega|F) = \underline{P}(\Omega|F^c) = \underline{P}(\Omega)$,

(s3) $\overline{P}(F|\Omega) = \overline{P}(F)$,

$\underline{P}(F|\Omega) = \underline{P}(F)$.

Since conditions $(s2), (s3)$ are always satisfied then Ω and F are s-independent if and only if $dim_H(\Omega) = dim_H(F)$. $\diamond$

Remark 1 *The previous result permits to put in evidence dependence between events. In fact, according to the axiomatic definition of stochastic independence, given a probability space $(\Omega, \boldsymbol{F}, P)$ and the σ-field $\boldsymbol{G} = \{\oslash, \Omega\}$, if conditional probability is defined by the Radon-Nikodym derivative we have that $P(F|\boldsymbol{G}) = P(F)$ $\forall F \in \boldsymbol{F}$, that is any event F is independent from $\boldsymbol{G}$.*

If Ω is a finite set and F is a non-empty set $\in \wp(\Omega)$, Ω and F are s-independent since any non-empty finite set has Hausdorff dimension equal to 0.

We extend the notions of s-irrelevance and s-independence to risks. Let S be a subclass of $\wp(\Omega)$ closed under intersection. The *atoms* of S are the minimal sets with respect to inclusion in $S - \{\oslash\}$.

Definition 6 *Let $S = \{C_i\}_{i\in N}$ be a countable class, the constituents of S are the sets $C = \bigcap_{i\in N} \widetilde{C}_i$ where $\widetilde{C}_i = C_i$ or $\widetilde{C}_i = C_i^c$.*

Definition 7 *The partition $\boldsymbol{C}(S)$ generated by a countable class $S \subseteq \wp(\Omega)$ is the partition of its constituents.*

Definition 8 *The σ-field generated by a class $S \subseteq \wp(\Omega)$ is the smallest σ-field $\sigma(S)$ containing S.*

In [15, Proposition 4.30] the following result has been proved.

Proposition 1 *Let S be a countable class of subsets of Ω and $\boldsymbol{G}$ a σ-field such that the class of the constituents $\boldsymbol{C}(S) \subseteq \boldsymbol{G} \subseteq \sigma(S)$. The atoms of $\boldsymbol{G}$ are the constituents of S.*

Remark 2 *If (Ω, d) is the Euclidean metric space the σ-field generated by the class $\{[x, +\infty); x \in \Re\}$ is the Borel σ-field, which can be also generated by the countable class $\{[x, +\infty); x \in Q\}$. The class of the singletons $\{x\}$ in $\Re$ is contained in the Borel σ-field but it does not generate it. So the Borel σ-field is countably generated.*

Definition 9 *Let $X \in L(\Omega)$. The partition $\boldsymbol{B}(X)$ generated by the random variable X is the partition of the non-empty constituents generated by the countable class $S = \{X^{-1}[x, +\infty); x \in Q\}$. It is countably generated.*

Proposition 2 *We have that*

$$\boldsymbol{B}(X) = \left\{X^{-1}\{x\}; x \in \Re\right\} - \{\oslash\}.$$

Example 2 *Let $X = I_A$ be the indicator function of a set $A \subseteq \Omega$, then*

$$X^{-1}[x, +\infty) = \Omega \text{ if } x \leq 0,$$
$$X^{-1}[x, +\infty) = A \text{ if } 0 < x \leq 1,$$
$$X^{-1}[x, +\infty) = \oslash \text{ if } x > 1.$$

Thus $S = \left\{X^{-1}[x, +\infty); x \in \Re^+\right\} = \{\Omega, A, \oslash\}$ and the class of atoms $S(X)$ generated by the random variable X is $S(X) = \{A\}$. The partition generated by the random variable X is is $\boldsymbol{B}(X) = \boldsymbol{C}(S) - \{\oslash\} = \{A, A^c\}$

Definition 10 *Two classes of events S_1 and S_2 are s-independent if for every $E \in S_1$ and $F \in S_2$ the events E and F are s-independent.*

The class S_2 is s- irrelevant to the class S_1 if for every $E \in S_1$ and every $F \in \boldsymbol{S}_2$ F is s-irrelevant to E.

The class S_1 is s-irrelevant to the class S_2 if for every $E \in S_1$ and every $F \in S_2$ E is s-irrelevant to F.

Definition 11 *Given a risk X the class of the weak upper level sets of X is $\left\{X^{-1}\{x\}; x \in \Re\right\}$. Let X and $Y \in L(\Omega)$ be two risks and let $S(X)$ and $S(Y)$ be the classes of atoms generated respectively by the class of the weak upper level sets of the risks X and Y. Then*

- *X and Y are s-independent if $S(X)$ and $S(Y)$ are s-independent;*
- *Y is s-irrelevant to X if $S(Y)$ is s-irrelevant to $S(X)$;*
- *X is s-irrelevant to Y if $S(X)$ is s-irrelevant to $S(Y)$.*

Example 3 *Let $\Omega = [0, 1]$, let $E = [0; \frac{1}{3}]$, $E_1 = [\frac{2}{3}; 1]$ $E_2 = [\frac{1}{3}; \frac{2}{3}]$ and let X and $Y \in L(\Omega)$ be two risks defined by $Y(\omega) = K$ and $X(\omega) = 1$ if $\omega \in E$, $X(\omega) = 2$ if $\omega \in E_1$ and $X(\omega) = 0$ if $\omega \in E_2$.*

Then

$$X^{-1}[x, +\infty) = \Omega \text{ if } x \leq 0,$$
$$X^{-1}[x, +\infty) = E \cup E_1 \text{ if } 0 < x \leq 1,$$
$$X^{-1}[x, +\infty) = E_1 \text{ if } 1 < x \leq 2,$$
$$X^{-1}[x, +\infty) = \oslash \text{ if } x > 2.$$

So $S(X) = \{E_1\}$ and $S(Y) = \{\Omega\}$. By Theorem 3 E_1 and Ω are s-independent since $dim_H E_1 = dim_H \Omega = 1$ so that X and Y are s-independent.

Theorem 4 *Let $X = I_A$ and $Y = I_E$ be the indicator functions of two sets $A, E \subseteq \Omega$, then Y is s-irrelevant to X if and only if E is s-irrelevant to A.*

Proof. By Definition 11 Y is s-irrelevant to X if and only if $S(Y)$ is s-irrelevant to $S(X)$. Since $S(Y) = \{E\}$ and $S(X) = \{A\}$ then Y is s-irrelevant to X if and only if E is s-irrelevant to A.$\diamond$

Remark 3 *S-independence of the indicator functions of two events A and E does not imply s-independence of the indicator functions of their complements because the classes of atoms respectively generated by $X = I_A$ and $Y = I_E$ do not contain the sets A^c and E^c. It is the main difference with the standard definition of independence, according to which two random variables are independent if and only if the σ-field generated by them are independent. Since the σ-fields generated by I_A and by I_{A^c} are equal to $\sigma(X) = \{\Omega, A, A^c, \oslash\}$ and the σ-fields generated by I_E and I_{E^c} are equal to $\sigma(Y) = \{\Omega, E, E^c, \oslash\}$, independence of I_A and I_E implies that I_{A^c} and I_{E^c} are independent.*

In [9, Theorem 9] it has been proven that curves filling the space like Peano curve and Hilbert curve are s-independent, so that by Definition 5 and Theorem 4 the indicator functions of curves filling the space are s-independent.

Example 4 *Let $\Omega = [0,1]$ and let X and $Y \in L(\Omega)$ be two risks defined by $Y(\omega) = K$ and $X(\omega) = 1$ if $\omega \in Q \cap [0,1]$ and $X(\omega) = 0$ otherwise.*

Then

$$X^{-1}[x,+\infty) = \Omega \text{ if } x \leq 0,$$
$$X^{-1}[x,+\infty) = Q \cap [0,1] \text{ if } 0 < x \leq 1,$$
$$X^{-1}[x,+\infty) = \oslash \text{ if } x > 1.$$

We have that $S(X) = \{Q \cap [0,1]\}$ and $S(Y) = \{\Omega\}$. By Theorem 3 $Q \cap [0,1]$ and Ω are s-dependent since

$$dim_H Q \cap [0,1] = 0 \neq 1 = dim_H \Omega$$

and so X and Y are s-dependent.

According to the axiomatic definition of stochastic independence X and Y are independent since the σ-field $\mathbf{G}(Y)$ generated by Y is $\mathbf{G}(Y) = \{\Omega, \oslash\}$ so that $P[A|\mathbf{G}(Y)] = P(A)$ with probability 1 for every A belonging to the σ-field generated by X.

Theorem 5 *Let $X, Y \in L(\Omega)$ and let $\boldsymbol{B}(X)$ and $\boldsymbol{B}(Y)$ be the partitions generated by them. A sufficient condition for s-irrelevance of Y to X is that $\boldsymbol{B}(Y)$ is s-irrelevant to $\boldsymbol{B}(X)$.*

Proof. The class of atoms is contained in the partition generated by a random variable, that is $S(X) \subseteq \mathbf{B}(X)$ and $S(Y) \subseteq \mathbf{B}(Y)$ $\forall X, Y \in L(\Omega)$, so if $\mathbf{B}(Y)$ is s-irrelevant to $\mathbf{B}(X)$ it implies that $S(Y)$ is s-irrelevant to $S(X)$. $\diamond$

Theorem 6 *Let $X, Y \in L(\Omega)$. A necessary condition for s-irrelevance of Y to X is that, for every $E \in \boldsymbol{S}(X)$, for every $F \in \boldsymbol{S}(Y)$ and $\omega \in F$ the following equality holds*

$$\overline{P}(E|\boldsymbol{B}(Y))(\omega) = \overline{P}(E|F) = \overline{P}(E)$$

Proof. Let Y be s-irrelevant to X; then by conditions s2) and s3) of Definition 4 we have that $\overline{P}(E|F) = \overline{P}(E)$ for every $E \in \mathbf{S}(X)$ and $F \in \mathbf{S}(Y)$, that is for $\omega \in \Omega$ with $\omega \in F$

$$\overline{P}(E|\mathbf{B}(Y))(\omega) = \overline{P}(E|F) = \overline{P}(E). \quad \diamond$$

Lemma 1 *Let $X, Y \in L(\Omega)$ be two risks such that X is $\boldsymbol{B}(Y)$-measurable then $\forall E \in S(X)$ we have that I_E is $\boldsymbol{B}(Y)$-measurable.*

Proof. Since X is $\mathbf{B(Y)}$-measurable, the sets of the partition $\mathbf{B}(X) = \{X^{-1}(x); x \in \Re\}$ are union of sets belonging to $\mathbf{B(Y)}$ and for every $E \in \mathbf{B}(X)$ the indicator function I_E is $\mathbf{B}(Y)$-measurable. Moreover $S(X) \subseteq \mathbf{B}(X)$ so the Lemma is proven.$\diamond$

Theorem 7 *Let $X, Y \in L(\Omega)$ such that X is $\boldsymbol{B}(Y)$-measurable and $S(X) \neq \{\Omega\}$ and $S(Y) \neq \{\Omega\}$. Then Y is s-relevant to X.*

Proof. Since X is $\boldsymbol{B}(Y)$-measurable by the coherence of $\overline{P}$ [21, property (f), p. 292] we have $\overline{P}(X|\boldsymbol{B}(Y)) = X$.

Let $B \in \mathbf{B}(Y)$, for every $\omega \in \Omega$ with $\omega \in B$ we have

$$\overline{P}(X|\boldsymbol{B}(Y))(\omega) = \overline{P}(X|B) = X(\omega).$$

Since X is $\mathbf{B}(Y)$-measurable by Lemma 1, for every $E \in \mathbf{B}(X)$ the indicator function I_E is $\mathbf{B}(Y)$-measurable so that

$$\overline{P}(I_E|\boldsymbol{B}(Y)) = I_E$$

so, since $S(X) \subseteq \mathbf{B}(X)$ and $S(X) \neq \{\Omega\}$, the necessary condition for s-irrelevance of Y to X given in Theorem 6 is not satisfied and Y is s-relevant to X.$\diamond$

The previous theorem does not hold if $S(X) = \{\Omega\}$ or $S(Y) = \{\Omega\}$.

Example 5 *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and let $X, Y \in L(\Omega)$ such that $X(\omega_1) = 1, X(\omega_2) = 2, X(\omega_3) = 3$ and $Y(\omega_i) = 1$ for $i = 1, ..., 3$. Then*

$$X^{-1}[x,+\infty) = \Omega \text{ if } x \leq 1,$$
$$X^{-1}[x,+\infty) = \{\omega_2, \omega_3\} \text{ if } 1 < x \leq 2,$$
$$X^{-1}[x,+\infty) = \{\omega_3\} \text{ if } 2 < x \leq 3,$$
$$X^{-1}[x,+\infty) = \oslash \text{ if } x > 3.$$

and

$$Y^{-1}[x,+\infty)=\Omega \text{ if } x\leq 1,$$
$$Y^{-1}[x,+\infty)=\oslash \text{ if } x>1.$$

So we have that $S(X)=\{\omega_3\}$ and $S(Y)=\{\Omega\}$ and by Definition 11 and Theorem 3 X and Y are s-independent since $dim_H\{\omega_3\}=0=dim_H\Omega$.

Proposition 3 *Let $X,Y\in L(\Omega)$ such that $X(\omega)=k$ and $Y(\omega)=h$ for every $\omega\in\Omega$ Then X and Y are s-independent.*

Proof. X and Y are constant thus $S(X)=S(Y)=\{\Omega\}$ and by Definition 11 X and Y are s-independent.$\diamond$

Example 6 *Let $X,Y\in L(\Omega)$ such that $\boldsymbol{B}(Y)$ is the partition of singletons of Ω and $S(X)\neq\{\Omega\}$. Then Y is s-relevant to X since X is $\boldsymbol{B}(Y)$-measurable and from Theorem 7 we have that Y is s-relevant to X.*

4 Dependent Risks

In this section sufficient conditions for s-dependence for risks are given. Surjective strictly monotone risks are proven to be s-dependent and every risk is proven to be s-dependent on a bijective risk.

Theorem 8 *Let X and $Y\in L(\Omega)$ be two risks and let $S(X)$ and $S(Y)$ be the classes of atoms generated respectively by the class of the weak upper level sets of the risks X and Y. If there exist $A\in S(X)$ and $F\in S(Y)$ such that $dim_H A\neq dim_H F$ then X and Y are s-dependent.*

Proof. Let $A\in S(X)$ and $F\in S(Y)$ such that $dim_H A\neq dim_H F$ so A and F are s-dependent since condition $s1)$ of Definition 5 is not satisfied. Thus the classes $S(X)$ and $S(Y)$ are s-dependent and by Definition 10 X and Y are s-dependent. $\diamond$

Example 7 *Let $([0,1],d)$ be the Euclidean metric space and let $X,Y\in L(\Omega)$ be two risks defined by $X(\omega)=1$ if $0<\omega<\frac{1}{2}, X(\omega)=2$ if $\omega=\frac{1}{2}$ and $X(\omega)=0$ otherwise, $Y(\omega)=1$ if $0<\omega<\frac{1}{2}$, $Y(\omega)=\frac{1}{2}$ if $\omega\geq\frac{1}{2}$;*

So we have

$$X^{-1}[x,+\infty)=\Omega \text{ if } x\leq 0,$$
$$X^{-1}[x,+\infty)=[0,\frac{1}{2}] \text{ if } 0<x\leq 1,$$
$$X^{-1}[x,+\infty)=\left\{\frac{1}{2}\right\} \text{ if } 1<x\leq 2,$$
$$X^{-1}[x,+\infty)=\oslash \text{ if } x>2,$$

and

$$Y^{-1}[x,+\infty)=\Omega \text{ if } x\leq\frac{1}{2},$$
$$Y^{-1}[x,+\infty)=[0,\frac{1}{2}] \text{ if } \frac{1}{2}<x\leq 1,$$
$$Y^{-1}[x,+\infty)=\oslash \text{ if } x>1.$$

Thus $S(X)=\left\{\{\frac{1}{2}\}\right\}$ and $S(Y)=\left\{[0,\frac{1}{2}]\right\}$ and by Theorem 8 the risks X and Y are s-dependent. We can observe that the events belonging to $S(X)$ and $S(Y)$ satisfy the factorization property with respect to μ^*_Ω, which is the Lebesgue measure h^1 since the Hausdorff dimension of $\Omega=[0,1]$ is 1. In fact the following equalities hold:

$$h^1(\left\{\frac{1}{2}\right\})h^1([0,\frac{1}{2}])=0=h^1(\left\{\frac{1}{2}\right\}\cap[0,\frac{1}{2}]).$$

Theorem 9 *Let $([0,1],d)$ be the Euclidean metric space and let $X,Y\in L(\Omega)$ be two risks such that Y is bijective. Then X ad Y are s-dependent.*

Proof. Since Y is bijective, the partition generated by Y is $\mathbf{B}(Y)=\left\{Y^{-1}\{x\};x\in\Re\right\}-\{\oslash\}$, that is the partition of singletons of $[0,1]$. So for any risk X condition $(s1)$ of Definition 5 is not satisfied for every $F\in S(X)$, then we have that X and Y are s-dependent. $\diamond$

Corollary 2 *Let $([0,1],d)$ be the Euclidean metric space and let $X,Y\in L(\Omega)$ be two surjective and strictly monotone random variables. Then X and Y are s-dependent.*

Proof. Since X and Y are surjective and strictly monotone the partition generated by them is the partition of singletons of [0,1]; from the fact that strictly monotonicity implies injectivity by Theorem 9 X is s-relevant to Y and Y is s-relevant to X; so X and Y are s-dependent.$\diamond$

Theorem 10 *Let $\Omega=\{\omega_1,\omega_2,...,\omega_n\}$ and let $X,Y\in L(\Omega)$ be two risks such that $Y:\{\omega_1,\omega_2,...,\omega_n\}\to\{x_1,x_2,...,x_n\}$ is injective. Then X and Y are s-dependent.*

Proof. Since Y is surjective if and only if it is injective, so the partition generated by Y is $\mathbf{B}(Y)=\left\{Y^{-1}\{x\};x\in\Re\right\}-\{\oslash\}$, that is the partition of singletons of Ω. So for any risk X condition $(s1)$ of Definition 5 is not satisfied for every $F\in S(X)$, then we have that X and Y are s-dependent. $\diamond$

5 Joint Coherent Conditional Measure of Risk

Let X and Y be two risks belonging to $L(\Omega)$. In this section the joint measure of risk $\rho(X,Y)$ of X and Y is defined and some properties are proven.

Definition 12 *Let X and Y be two risks belonging to $L(\Omega)$ and let $\boldsymbol{B}(X)$ be the partition generated by X. The coherent upper conditional prevision of Y given X, denoted by $\overline{P}(Y|X)(\omega)$ is the random variable on Ω defined by*

$$\overline{P}(Y|X)(\omega) = \overline{P}(Y|\boldsymbol{B}(X)) = \overline{P}(Y|B)$$

if $\omega \in B$ and $B \in \boldsymbol{B}(X)$.

Moreover if B has positive and finite Hausdorff outer measure in its Hausdorff dimension s the coherent conditional measure of risk $\rho(Y|X)$ of Y given X is defined by

$$\rho(Y|X) = \overline{P}(Y|\boldsymbol{B}(X)) = \overline{P}(Y|B) = \int_B Y d\mu_B^*.$$

If Y is $\mathbf{B}(X)$-measurable (i.e. it is constant on the sets of $\mathbf{B}(X)$) then $\overline{P}(Y|X) = Y$.

Given a risk X let $A = X^{-1}(A')$ be the inverse image of A' for every Borelian set A' of $\Re$.

Definition 13 *Given a risk $X \in L(\Omega)$ and denoted by t the Hausdorff dimension of Ω, the coherent upper probability $\overline{P}_X$ induced by X on $\Re$ is defined by*

$$\overline{P}_X(A') = \overline{P}(A) = \frac{h^t\{\omega \in \Omega : X(\omega) \in A'\}}{h^t(\Omega)}$$

for every Borelian set A' of $\Re$.

Definition 14 *Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension t. Given two risks $X, Y \in L(\Omega)$ the joint coherent upper probability $\overline{P}_{(X,Y)}$ induced by the pair (X,Y) on $\Re \times \Re$ is defined by*

$$\begin{aligned}\overline{P}_{(X,Y)}&(A' \times B') \\ &= \overline{P}(A \cap B) \\ &= \frac{h^t\{\omega \in \Omega : (X(\omega), Y(\omega)) \in A' \times B'\}}{h^t(\Omega)}\end{aligned}$$

for every pair of Borelian sets A' and B' of $\Re$.

Definition 15 *Given two risks $X, Y \in L(\Omega)$ such that the Hausdorff dimension of the inverse image $A = X^{-1}(A')$ is s and $0 < h^s(A) < +\infty$ the joint coherent upper probability $\overline{P}_{(X,Y)|X}$ given X is defined by*

$$\begin{aligned}\overline{P}_{(X,Y)|X}&((A' \times B') \times A') \\ &= \overline{P}((A \cap B)|A) \\ &= \frac{h^s\{\omega \in \Omega : (X(\omega), Y(\omega)) \in A' \times B'\}}{h^s(A)}\end{aligned}$$

for every pair of Borelian sets A' and B' of $\Re$.

Definition 16 *Given two risks $X, Y : \Omega \longrightarrow \Re$ and a partition $\boldsymbol{B}$ of Ω let $B \in \boldsymbol{B}$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s; the joint coherent conditional measure of risk $\rho((X,Y)|B)$ is defined by*

$$\begin{aligned}\rho(&(X,Y)|B) \\ &= \int_B \overline{P}_{(X,Y)} d\mu_B^* \\ &= \int \frac{h^s\{\omega \in B : (X(\omega), Y(\omega)) \in A' \times B'\}}{h^s(B)} dx.\end{aligned}$$

6 Properties of s-Independent Risks

In this section the relation between s-independence and the factorization of the joint distribution $\overline{P}_{(X,Y)}$ into the product of the marginal distributions $\overline{P}_X$ and $\overline{P}_Y$ is investigated.

According to the axiomatic definition two random variables X and Y are independent if and only if $\sigma(X)$ and $\sigma(Y)$, the σ-fields generated by them, are independent. Since the σ-field generated by a random variable X is the smallest σ-field with respect to which X is measurable, it contains the inverse image of all Borelian sets of $\Re$. So if the random variables X and Y are independent then the joint distribution is equal to the product of the marginal distributions.

S-independence of risk X and Y does not imply that the joint distribution $\overline{P}_{(X,Y)}$ is equal to the product of the marginal $\overline{P}_X$ and $\overline{P}_Y$. It occurs because s-independence between risks implies that the factorization property holds only for the atoms (i.e. minimal ses with respect to the inclusion) of the classes generated by the weak upper level sets of X and Y (see Corollary 1) and not for all sets of the σ-fields generated by X and Y.

In the next theorem a sufficient condition is given to assure that the joint distribution of two simple risks is equal to the product of their marginal distributions.

Theorem 11 *Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension and let X and $Y \in L(\Omega)$ be two simple risks such* *that the partition $\boldsymbol{B}(Y)$ is s-irrelevant to the partition $\boldsymbol{B}(X)$. Then*

$$\rho(XY) = \overline{P}(XY) = \overline{P}(X)\overline{P}(Y) = \rho(X)\rho(Y).$$

Proof. Let X and Y be simple risks. Let $A_1, ..., A_n$ the atoms of $\mathbf{B}(X)$ and $B_1, ..., B_m$ the atoms of $\mathbf{B}(Y)$ where the atoms A_i and B_j are enumerated so that $x_i = X(A_i)$ for $i = 1, ..., n$ and $y_j = Y(B_j)$ for $j = 1, ...m$ are in descending order, i.e. $x_1 \geq x_2 \geq ... \geq x_n$ with $x_{n+1} = 0$ and $y_1 \geq y_2 \geq ... \geq y_m$ with $y_{m+1} = 0$. Thus $X = \sum_{i=1}^{n} x_i I_{A_i}$ and $Y = \sum_{j=1}^{m} x_j I_{B_j}$ and since $\mathbf{B}(Y)$ is s-irrelevant to $\mathbf{B}(X)$ by Corollary 1 we have that

$$\mu^*_{\Omega}(A_i \cap B_j) = \mu^*_{\Omega}(A_i)\mu^*_{\Omega}(B_j).$$

The coherent upper probability μ^*_{Ω} is submodular and since the random variable $Z = XY$ and any constant c are comonotonic we consider the class $\mathbf{C} = \{XY, c\}$ so that by Proposition 10.1 of [5] there exists an additive set function α on $\wp(\Omega)$ which agrees with μ^*_{Ω} on the class of μ^*_{Ω}-measurable sets (and so on the atoms of $\mathbf{B}(X)$ and on the atoms of $\mathbf{B}(Y)$) such that

$$\begin{aligned}\int_{\Omega} XY d\mu^*_{\Omega} &= \int_{\Omega} XY d\alpha \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \alpha(I_{A_i} I_{B_j}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \mu^*_{\Omega}(I_{A_i} I_{B_j}).\end{aligned}$$

Thus the following equalities hold

$$\begin{aligned}\rho(XY) &= \overline{P}(XY) = \int XY d\mu^*_{\Omega} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \mu^*_{\Omega}(I_{A_i} I_{B_j}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \mu^*_{\Omega}(A_i \cap B_j) \\ &= \sum_{i=1}^{n} x_i \mu^*_{\Omega}(A_i) \sum_{J=1}^{m} y_j \mu^*_{\Omega}(B_j) \\ &= \overline{P}(X)\overline{P}(Y) = \rho(X)\rho(Y). \quad \diamond\end{aligned}$$

Corollary 3 *Let $B \in \boldsymbol{B}$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension and let $X|B$ and $Y|B \in L(B)$ be two simple risks such that $\boldsymbol{B}(Y|B)$ is s-irrelevant to $\boldsymbol{B}(X|B)$. Then*

$$\begin{aligned}\rho(XY|B) &= \overline{P}(XY|B) \\ &= \overline{P}(X|B) \cdot \overline{P}(Y|B) = \rho(X|B)\rho(Y|B).\end{aligned}$$

The next example shows that s-independence of two risks X and Y does not imply s-independence of $X|B$ and $Y|B$ where B is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension belonging to a partition $\mathbf{B}$ of Ω.

Example 8 *Let X and $Y \in L(\Omega)$ be the two s-independent risks defined in Example 3 and let $B = [0, \frac{2}{3}]$. Since $S(X|B) = \left\{\frac{2}{3}\right\}$ and $S(Y|B) = B$ then by Theorem 8 $X|B$ and $Y|B$ are s-dependent.*

7 Conclusions

Two risks X and Y are s-independent if the atoms of the classes generated by their weak upper level sets are s-independent. A crucial difference with respect to the axiomatic definition of independence for random variables is that s-independence of the indicator functions of two events does not imply s-independence of the indicator functions of their complements. Moreover s-independence for risks does not imply the factorization of the joint distribution into the product of the marginal distributions. In the model, different experiences of two decision makers are represented by different conditioning events with different Hausdorff dimension and this produces two different measures of risk for the same random variable. This allows us to mathematically represent the fact that different decision makers can retain certain actions more risky or less risky according to their own experiences or the information that they hold. That is to say what is for one a disadvantage for another person could be considered a convenient choice .

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