Weak Consistency for Imprecise Conditional Previsions

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Outline

Main purpose: introduce general consistency concepts for conditional lower previsions (CLP), weaker than (Williams-)coherence and convexity, preserving a unitary approach.

Starting point: generalise n-coherent unconditional previsions in Walley (1991).

Focus on (centered) 2-convex and 2-coherent CLP, as these are the *most significant and general* models within n-convexity and n-coherence, and study their characterisation and main properties.

Characterise 2-convexity and 2-coherence in terms of *desirability*.

Coherence and convexity

Let $\underline{P}: \mathcal{D} \to \mathbb{R}$ be a conditional lower prevision. $\forall n \in \mathbb{N}_0$, $X_0|B_0, \dots, X_n|B_n \in \mathcal{D}$, $s_0 \in \mathbb{R}$, $s_1, \dots, s_n \geq 0$ define

$$S(\underline{s}) = \bigvee\{B_i : s_i \neq 0, i = 0, \dots, n\},$$
$$\underline{G} = \sum_{i=1}^n s_i B_i (X_i - \underline{P}(X_i|B_i)) - s_0 B_0 (X_0 - \underline{P}(X_0|B_0)),$$

$\forall \underline{G}$ s.t. $S(\underline{s}) \neq \varnothing$, let $\sup\{\underline{G}|S(\underline{s})\} \geq 0$.

- $s_0 \geq 0 \Rightarrow \underline{P}$ is *coherent* (Williams, 1975).
- $\sum_{i=1}^n s_i = 1 = s_0$ (convexity constraint) $\Rightarrow \underline{P}$ is *convex* (Pelessoni, Vicig, 2005)
- Convexity + $0|B \in \mathcal{D}$ and $\underline{P}(0|B) = 0$, $\forall X|B \in \mathcal{D} \Rightarrow \underline{P}$ *centered convex (C-convex)*

Weakening coherence/convexity

Basic idea: introduce constraints on n.
Most prominent case: $n = 1$ (2 addends in $\underline{G}$).

$$\sup\{\underline{G}|S(\underline{s})\} \geq 0 \text{ (with } S(\underline{s}) \neq \varnothing)$$

- $\forall \underline{G}$ s.t. $n = 1 \Rightarrow \underline{P}$ is *2-coherent*.
- $\forall \underline{G}$ s.t. $n = 1$, $s_0 = s_1 = 1$, $\Rightarrow \underline{P}$ is *2-convex*.
- 2-convexity + $0|B \in \mathcal{D}$ and $\underline{P}(0|B) = 0$, $\forall X|B \in \mathcal{D} \Rightarrow \underline{P}$ is *centered 2-convex* .

Features of 2-convexity

Let $\underline{P}: \mathcal{D} \to \mathbb{R}$ be a 2-convex CLP.

- *Positive homogeneity* and *internality* ($\underline{P}(X|B) \in [\inf X|B, \sup X|B]\ \forall X|B \in \mathcal{D}$) do not necessarily hold.
- Non-internality cannot be two-sided.
- If centered , $\underline{P}$ satisfies internality, has a 2-convex natural extension, and agrees with the Goodman-Nguyen relation (= conditional implication).

Properties of 2-coherence

Additional properties of a 2-coherent $\underline{P}: \mathcal{D} \to \mathbb{R}$ with respect to centered 2-convexity.

- $\underline{P}(X|B) \leq \overline{P}(X|B) = -\underline{P}(-X|B)$
- $\underline{P}$ is positively homogeneous.
- $\underline{P}$ has a 2-coherent natural extension.

The structured set $\mathcal{D}_{LIN}$

Let $\mathcal{X}$ be a linear space of gambles, $\mathcal{B} \subset \mathcal{X}$ the set of all events.
Let also $1 \in \mathcal{B}$, $BX \in \mathcal{X}, \forall B \in \mathcal{B}, \forall X \in \mathcal{X}$, $\mathcal{B}^{\varnothing} = \mathcal{B} - \{\varnothing\}$ and define

$$\mathcal{D}_{LIN} = \{X|B : X \in \mathcal{X}, B \in \mathcal{B}^{\varnothing}\}.$$

Coherence, convexity, 2-coherence and 2-convexity can be characterised on $\mathcal{D}_{LIN}$ through *sets of axioms*.

Some lower prevision axioms

(DI) $\underline{P}(X|B) - \underline{P}(Y|B) \leq \sup\{X - Y|B\}$, $\forall X|B, Y|B \in \mathcal{D}_{LIN}$.
(*Difference Internality*.)

(GBR) $\underline{P}(A(X - \underline{P}(X|A \wedge B))|B) = 0$, $\forall X \in \mathcal{X}, \forall A, B \in \mathcal{B}^{\varnothing} : A \wedge B \neq \varnothing$.
(*Generalised Bayes Rule*.)

(PH) $\underline{P}(\lambda X|B) = \lambda \underline{P}(X|B)$, $\forall X|B \in \mathcal{D}_{LIN}, \forall \lambda \geq 0$.
(*Positive Homogeneity*.)

(NWH) $\underline{P}(\lambda X|B) \leq \lambda \underline{P}(X|B), \forall \lambda < 0$
(*Negative Weak Homogeneity*.)

Characterisation through axioms

On $\mathcal{D}_{LIN}$,

- $\underline{P}$ is 2-*convex* iff (DI) and (GBR) hold.
- $\underline{P}$ is 2-*coherent* iff (DI), (GBR), (PH) and (NWH) hold.

Remark: n-convex (n-coherent) lower previsions ($n \geq 3$) either are convex (coherent) themselves or have no n-convex (n-coherent) natural extension on any set.

A desirability approach

Given $\mathcal{D}_{LIN}$, define

$$\mathcal{X}^{\succeq} = \{X \in \mathcal{X} : \inf X \geq 0\},$$
$$\mathcal{X}^{\preceq} = \{X \in \mathcal{X} : \sup X \leq 0\},$$

and, $\forall B \in \mathcal{B}$,

$$\mathcal{R}(B) = \{X \in \mathcal{X} : BX = X\},$$
$$\mathcal{R}(B)^{\succ} = \{X \in \mathcal{R}(B) : \inf\{X|B\} > 0\},$$
$$\mathcal{R}(B)^{\prec} = \{X \in \mathcal{R}(B) : \sup\{X|B\} < 0\}.$$

Williams (1975, 2007) characterised conditional coherence through desirability axioms.

Acceptable gambles - 1

Define, $\forall X|B \in \mathcal{D}_{LIN}$,

$$\underline{P}(X|B) = \sup\{x \in \mathbb{R} : B(X - x) \in \mathcal{A}\}.$$

where $\mathcal{A} \subseteq \mathcal{X}$ is a set of acceptable gambles.

a) $\lambda\mathcal{A} + \mathcal{R}(B)^{\succ} \subseteq \mathcal{A}, \forall \lambda \geq 0, B \in \mathcal{B}$;
b) $\mathcal{R}(B)^{\prec} \cap \mathcal{A} = \emptyset, \forall B \in \mathcal{B}$;
c) $(\mathcal{R}(B_1) \cap \mathcal{A}) + (\mathcal{R}(B_2) \cap \mathcal{A}) \subseteq \mathcal{R}(B_1 \vee B_2) \setminus \mathcal{R}(B_1 \vee B_2)^{\prec}, \forall B_1, B_2 \in \mathcal{B}$.

$\Rightarrow \underline{P}$ is 2-coherent on $\mathcal{D}_{LIN}$.

a') $\mathcal{A} + \mathcal{R}(B)^{\succ} \subseteq \mathcal{A}, \forall B \in \mathcal{B}$;
b) $\mathcal{R}(B)^{\prec} \cap \mathcal{A} = \emptyset, \forall B \in \mathcal{B}$.

$\Rightarrow \underline{P}$ is 2-convex on $\mathcal{D}_{LIN}$;
$\underline{P}$ is centered iff $\mathcal{R}(B)^{\succ} \subseteq \mathcal{A}$, $\forall B \in \mathcal{B}$.

Comments

- a) and a') replace the cone conditions in (Williams, 1975)
- b) is a condition of avoiding partial loss
- c) means that: $X_i \in \mathcal{A}, B_i X_i = X_i$ $(i = 1, 2)$ $\Rightarrow \sup(X_1 + X_2|B_1 \vee B_2) \geq 0$, i.e. $X_1 + X_2$ may be not accepted, but is not necessarily discarded (by b) with $B = B_1 \vee B_2$)
- If $\underline{P}$ is not centered, some $X|B$ s.t. $\inf(X|B) > 0$ might be not acceptable!

Acceptable gambles - 2

Let $\underline{P}: \mathcal{D}_{LIN} \to \mathbb{R}$ be a CLP. Define

$$\mathcal{A}' = \{\lambda B(X - x) + Y : X|B \in \mathcal{D}_{LIN}, x < \underline{P}(X|B), Y \in \mathcal{X}^{\succeq}, \lambda \geq 0\};$$
$$\mathcal{A}'' = \{B(X - x) + Y : X|B \in \mathcal{D}_{LIN}, x < \underline{P}(X|B), Y \in \mathcal{X}^{\succeq}\}.$$

If $\underline{P}$ is 2-coherent,

a) $a\mathcal{A}' + \mathcal{X}^{\succeq} \subseteq \mathcal{A}', \forall a \geq 0$;
b) $\mathcal{X}^{\preceq} \cap \mathcal{A}' = \{0\}$;
c) $(\mathcal{A}' + \mathcal{A}') \setminus \{0\} \subseteq \mathcal{X} \setminus \mathcal{X}^{\preceq}$;
d) $\underline{P}(X|B) = \sup\{x \in \mathbb{R} : B(X - x) \in \mathcal{A}'\}$.

If $\underline{P}$ is 2-convex,

a') $\mathcal{A}'' + \mathcal{X}^{\succeq} \subseteq \mathcal{A}''$;
b') $\mathcal{X}^{\preceq} \cap \mathcal{A}'' = \emptyset$ iff $\underline{P}$ is 1-AUL;
d') $\underline{P}(X|B) = \sup\{x \in \mathbb{R} : B(X - x) \in \mathcal{A}''\}$;
e') $\underline{P}$ is centered iff $\mathcal{R}(B)^{\succ} \subseteq \mathcal{A}''$, $\forall B \in \mathcal{B}$.

2-convex and 2-coherent models

Let $I\!P$ be a finite partition, $\mathcal{L}$ be a linear space of random variables.

(Normalized) *capacity*: $c : 2^{I\!P} \to [0, 1]$ s.t.
- $c(\varnothing) = 0, c(\Omega) = 1$;
- $A_1 \Rightarrow A_2$ implies $c(A_1) \leq c(A_2)$.

Niveloid (Dolecki, Greco, 1995): $N : \mathcal{L} \to \overline{\mathbb{R}}$ s.t.
- $N(X + \mu) = N(X) + \mu, \forall X \in \mathcal{L}, \forall \mu \in \mathbb{R}$;
- $X \geq Y \Rightarrow N(X) \geq N(Y), \forall X, Y \in \mathcal{L}$.

Niveloids are not necessarily centered.

Proposition (Baroni, Pelessoni, Vicig, 2009)
a) $\underline{P} : 2^{I\!P} \to \mathbb{R}$ is a centered 2-convex lower prevision iff it is a capacity
b) $P : \mathcal{L} \to \mathbb{R}$ ($\mathcal{L}$ linear space of gambles) is a 2-convex lower prevision iff it is a (finite-valued) niveloid.

$\Rightarrow$ 2-convex conditional lower previsions can define conditional capacities and niveloids.

Using conjugate couples, like $(\underline{c}, \overline{c})$, we need 2-coherence to ensure $\underline{c} \leq \overline{c}$ (cf. also the case of bivariate p-boxes in (Pelessoni, Vicig, Montes, Miranda, submitted).)

Ways to incoherence... ;-)

(First image when googling 'incoherence')