Conformity and independence for coherent lower previsions

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Introduction

- Conformity of a marginal and a conditional lower prevision means that they can be derived from some joint by means of natural extension.
- We study this notion, also together with assumptions of epistemic irrelevance and independence between the variables.
- The connection with independent products, and in particular with the strong product, is also investigated.
- Settings: two variables X_1, X_2 taking values in possibility spaces $\mathcal{X}_1, \mathcal{X}_2$ (not necessarily finite).

BASIC NOTIONS:

Let $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2) := \{f : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R} \text{ bounded}\}$.

$\underline{P} : \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2) \to \mathbb{R}$ is a coherent lower prevision when it is the lower envelope of a family of expectations with respect to finitely additive probabilities.

Its restrictions to $\mathcal{X}_1, \mathcal{X}_2$—-measurable gambles are its marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$.

Similarly, given $x_1 \in \mathcal{X}_1$, $\underline{P}(\cdot|x_1) : \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2) \to \mathbb{R}$ is a conditional coherent lower prevision when it is the lower envelope of a family of conditional expectations with respect to finitely additive probabilities.

In that case, $\underline{P}(\cdot|\mathcal{X}_1) := \sum_{x_1 \in \mathcal{X}_1} I_{x_1} \cdot \underline{P}(\cdot|x_1)$ is a separately coherent conditional lower prevision.

Results

Preliminary concepts

Conditional natural extension: given $\underline{P}$ on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ and $x_1 \in \mathcal{X}_1$, $\underline{E}(f|x_1)$ is given by

$$\begin{cases} \sup\{\mu : \underline{P}(I_{x_1}(f-\mu)) \geq 0\} & \text{if } \underline{P}(x_1) > 0 \\ \inf_{x \in \mathcal{X}_2} f(x_1, x) & \text{otherwise.} \end{cases}$$

$\underline{E}(\cdot|\mathcal{X}_1)$ satisfies irrelevance w.r.t $\underline{P}_{\mathcal{X}_2}$ when

$$\underline{E}(f|x_1) = \underline{P}_{\mathcal{X}_2}(f(x_1, \cdot))$$

for all $f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2), x_1 \in \mathcal{X}_1$.

$\underline{E}(\cdot|\mathcal{X}_1), \underline{E}(\cdot|\mathcal{X}_2)$ satisfy irrelevance w.r.t $\underline{P}_{\mathcal{X}_2}, \underline{P}_{\mathcal{X}_1}$ when

$$\underline{E}(f|x_1) = \underline{P}_{\mathcal{X}_2}(f(x_1, \cdot))$$
$$\underline{E}(f|x_2) = \underline{P}_{\mathcal{X}_1}(f(\cdot, x_2))$$

for all $f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2), x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2$.

$\underline{P}$ is an independent product of the marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ when $\underline{P}, \underline{P}(\cdot|\mathcal{X}_1), \underline{P}(\cdot|\mathcal{X}_2)$ are coherent, where $\underline{P}(\cdot|\mathcal{X}_1), \underline{P}(\cdot|\mathcal{X}_2)$ are defined by epistemic irrelevance. The smallest one is the independent natural extension $\underline{P}_{\mathcal{X}_1} \otimes \underline{P}_{\mathcal{X}_2}$.

An independent product is an independent envelope of its marginals when it is a lower envelope of factorising linear previsions. The smallest is the strong product $\underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$, given by

$$\min\{P_1 \times P_2 : P_1 \geq \underline{P}_{\mathcal{X}_1}, P_2 \geq \underline{P}_{\mathcal{X}_2}\}.$$

Definitions

$\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$ are conforming when there exists $\underline{P}$ on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ with marginal $\underline{P}_{\mathcal{X}_1}$ and conditional natural extension $\underline{P}(\cdot|\mathcal{X}_1)$.

↓

$\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance when there exists $\underline{P}$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ and whose conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$ satisfies irrelevance w.r.t $\underline{P}_{\mathcal{X}_2}$.

↓

We say that $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ independence when there exists $\underline{P}$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ whose conditional natural extensions $\underline{E}(\cdot|\mathcal{X}_1), \underline{E}(\cdot|\mathcal{X}_2)$ satisfy irrelevance w.r.t. $\underline{P}_{\mathcal{X}_2}, \underline{P}_{\mathcal{X}_1}$.

Consider $\underline{P}$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$, and the conditions:

$$\underline{P}(f) \leq \underline{P}(P_{\mathcal{X}_2}(f|\mathcal{X}_1)) \quad (1)$$

and

$$\underline{P}(g-f) \geq \min_{x_1 \in \mathcal{X}_1} \underline{P}(g - f^{x_1}), \quad (2)$$

for all $f, g \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2), P_{\mathcal{X}_2} \geq \underline{P}_{\mathcal{X}_2}$, where

$$f^{x_1} : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}$$
$$(x_1', x_2) \mapsto f(x_1, x_2).$$

Results

- $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$ are conforming $\Leftrightarrow$ $\underline{P}(\cdot|x_1)$ is vacuous whenever $\underline{P}_{\mathcal{X}_1}(x_1) = 0$.
- If $\mathcal{X}_1$ is finite and there is some $\underline{P}$ conforming with $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$, the smallest one is $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$.

Let $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$ be the set joints with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ and satisfying conformity with irrelevance.

- $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})} \neq \emptyset \Leftrightarrow$ either $\underline{P}_{\mathcal{X}_1}(x_1) > 0 \ \forall x_1$ or $\underline{P}_{\mathcal{X}_2}$ is vacuous.
- $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$ is closed under lower envelopes.
- If $\mathcal{X}_1$ is finite and $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})} \neq \emptyset$, the smallest model in this set is $\underline{P}_{\mathcal{X}_1}(\underline{P}_{\mathcal{X}_2}) = \underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$, where $\underline{P}(\cdot|\mathcal{X}_1)$ is derived from $\underline{P}_{\mathcal{X}_2}$ by irrelevance.

Thus, conforming natural extension=irrelevant natural extension when $\mathcal{X}_1$ is finite (but not in general).

Let $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$ be the set of these compatible joints.

- $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})} \neq \emptyset \Leftrightarrow$ (a) either $\underline{P}_{\mathcal{X}_1}(x_1) > 0$ for every x_1 or $\underline{P}_{\mathcal{X}_2}$ is vacuous; and (b) either $\underline{P}_{\mathcal{X}_2}(x_2) > 0$ for every x_2 or $\underline{P}_{\mathcal{X}_1}$ is vacuous.
- If $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})} \neq \emptyset$, then any independent product of $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ belongs to $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2})}$.

When $\mathcal{X}_1, \mathcal{X}_2$ are finite:

- (1) $\Leftrightarrow$ $\underline{P} \leq \underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$.
- (2) $\Leftrightarrow$ $\underline{P}$ is an independent envelope.
- $\underline{P}$ satisfies (1) $\not\Rightarrow$ $\underline{P}$ independent product.
- $\underline{P} \geq \underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2} \not\Rightarrow \underline{P}$ satisfies (2).
- (1)+(2) $\Leftrightarrow$ $\underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$.

Essential references

- G. de Cooman, E. Miranda, M. Zaffalon, *Independent natural extension*. Artificial Intelligence, 2011.
- E. Miranda and M. Zaffalon, *Independent products in infinite spaces*. Journal of Mathematical Analysis and Applications, 2015.
- P. Walley, *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, 1991.
- M. Zaffalon and E. Miranda. *Desirability and the birth of incomplete preferences*. Submitted, 2015.

At a glance

Results:

- Conformity clashes with the existence of zero lower probabilities.
- In the finite case the conforming natural extension becomes the irrelevant/independent natural extension.
- We have a behavioural characterisation of the strong product.

Open problems:

- Study this problem with other updating rules, like regular extension.
- Extension to more than two models.
- Connection with sets of gambles.