Dilation, Disintegrations, and Delayed Decisions

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Abstract

Both dilation and non-conglomerability have been alleged to conflict with a fundamental principle of Bayesian methodology that we call *Good's Principle*: one should always delay making a terminal decision between alternative courses of action if given the opportunity to first learn, at zero cost, the outcome of an experiment relevant to the decision. In particular, both dilation and non-conglomerability have been alleged to permit or even mandate choosing to make a terminal decision in deliberate ignorance of relevant, cost-free information. Although dilation and non-conglomerability share some similarities, some authors maintain that there are important differences between the two that warrant endorsing different normative positions regarding dilation and non-conglomerability. This article reassesses the grounds for treating dilation and non-conglomerability differently. Our analysis exploits a new and general characterization result for dilation to draw a closer connection between dilation and non-conglomerability.

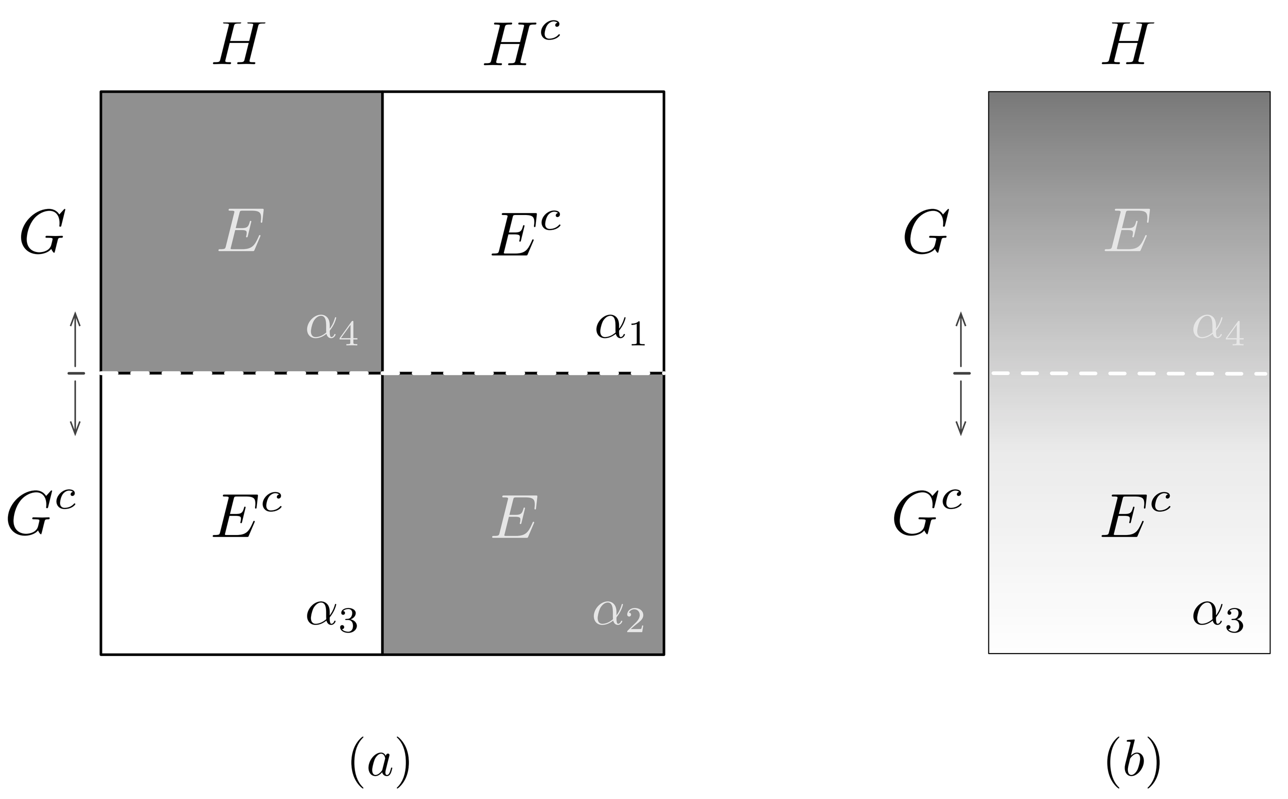
Dilation

Let $\mathfrak{E}$ be a positive measurable partition of Ω. Say that $\mathfrak{E}$ *dilates* an event E just in case for each $H \in \mathfrak{E}$:

$$\underline{P}(E \mid H) < \underline{P}(E) \leq \overline{P}(E) < \overline{P}(E \mid H).$$

In other words, the partition $\mathfrak{E}$ dilates E just in case the closed interval $\left[\underline{P}(E), \overline{P}(E)\right]$ is contained in the open interval $\left(\underline{P}(E|H), \overline{P}(E|H)\right)$ for each partition cell $H \in \mathfrak{E}$.

Dilation Example



$$\underline{P}(G) = 1/10 \qquad \overline{P}(G) = 9/10 \tag{1}$$

$$\underline{P}(H) = \overline{P}(H) = 1/2 = \underline{P}(H^c) = \overline{P}(H^c) \tag{2}$$

$$p(G \cap H) = p(G)p(H) = \frac{p(G)}{2} \tag{3}$$

$$1/10 = \underline{P}(E \mid H) < \underline{P}(E) = 1/2 = \overline{P}(E) < \overline{P}(E \mid H) = 9/10$$
$$1/10 = \underline{P}(E \mid H^c) < \underline{P}(E) = 1/2 = \overline{P}(E) < \overline{P}(E \mid H^c) = 9/10$$

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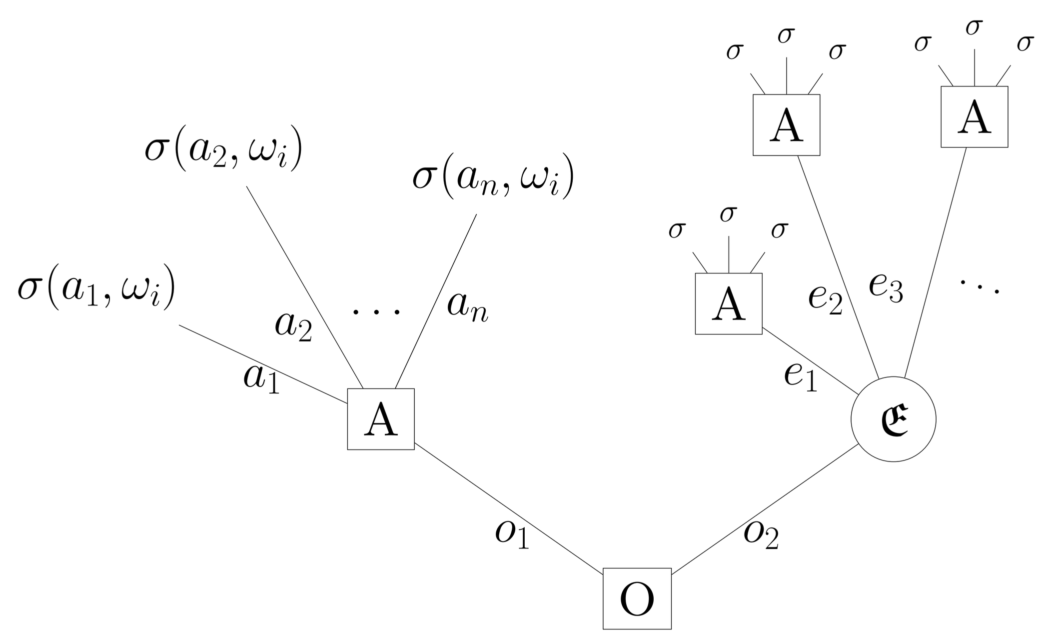
Sequential Decision Problem

Ω	Set of states ω
$\sigma(a,\omega)$	Consequence of a given ω



Suppose that at time t_1 you are to face a choice among several courses of action $A = \{a_1, \ldots, a_n\}$. Prior to this choice, however, you face a decision $O = \{o_1, o_2\}$ at some time t_0 before t_1 between (o_1) choosing from among the courses of action A at time t_1 or (o_2) choosing from among the courses of action A at some later time t_2 after you have observed, at no cost, the outcome e of an experiment $\mathfrak{E}$.

Good's Principle

(1) You are **prohibited** from rejecting at t_0 the opportunity to choose from among A at t_2 after observing the outcome of the experiment $\mathfrak{E}$.

(2) If the experimental outcome of $\mathfrak{E}$ might affect your choice from among A, then you are **prohibited** from deciding at t_0 to choose from among A at t_1.

Theorem

Let $(\Omega, \mathcal{A}, \mathbb{P}, \underline{P})$ be a lower probability space, let $\mathfrak{E} = \{H_i : i \in I\}$ be a positive measurable partition, and let $E \in \mathcal{A}$. Then the following are equivalent:

(i) $\mathfrak{E}$ dilates E;

(ii) There is $(\epsilon_i)_{i\in I} \in \mathbb{R}^I_+$ such that for every $i \in I$:

$$\mathbb{P}_*(E \mid H_i, \epsilon_i) \subseteq S^-_*(E, H_i) \quad \text{and} \quad \mathbb{P}_*(E \mid H_i, \epsilon_i) \subseteq S^+_*(E, H_i);$$

(iii) There is $(\epsilon_i)_{i\in I} \in \mathbb{R}^I_+$ such that for every $i \in I$:

$$\mathbb{P}(E \mid H_i, \epsilon_i) \subseteq S^-(E, H_i) \quad \text{and} \quad \mathbb{P}(E \mid H_i, \epsilon_i) \subseteq S^+(E, H_i),$$

where for each $i \in I$, $\epsilon_i \leq \min(\underline{\epsilon}_i, \overline{\epsilon}_i)$, and:

$\underline{\epsilon}_i$ is the unique minimum of $|p(E|H_i) - \underline{P}(E|H_i)|$ attained on $C^+_i =_{df} \{p \in \mathbb{P}^* : S_p(E, H_i) \geq 1\}$

$\overline{\epsilon}_i$ is the unique minimum of $|p(E|H_i) - \overline{P}(E|H_i)|$ attained on $C^-_i =_{df} \{p \in \mathbb{P}^* : S_p(E, H_i) \leq 1\}$. ⋄

Non-Conglomerability

Let $\mathfrak{E}$ be a positive measurable partition of Ω. Say p is *conglomerable* in $\mathfrak{E}$ for an event E if:

$$\inf\left\{p(E|H) : H \in \mathfrak{E}\right\} \leq p(E) \leq \sup\left\{p(E|H) : H \in \mathfrak{E}\right\}$$

Otherwise say that p is *non-conglomerable* in $\mathfrak{E}$ for E. So p is non-conglomerable in $\mathfrak{E}$ for E just in case $p(E)$ fails to lie in the closed interval $\left[\inf\{p(E|H) : H \in \mathfrak{E}\}, \sup\{p(E|H) : H \in \mathfrak{E}\}\right]$.

Non-Conglomerability Example

Let $\Omega = \{0,1\} \times \mathbb{N}_{>0}$, let $E = \{(1,n) : n \in \mathbb{N}_{>0}\}$, and let $\mathfrak{E} = \{H_n : n \in \mathbb{N}_{>0}\}$, where $H_n = \{(0,n), (1,n)\}$ for each $n \in \mathbb{N}_{>0}$.

$$p(E) = 1/2 \tag{4}$$

$$p(E \cap H_n) = 1/2^{n+1} \quad \text{for each } n \in \mathbb{N}_{>0} \tag{5}$$

$$p(E^c \cap H_n) = 0 \tag{6}$$

$$1/2 = p(E) < \inf\left\{p(E|H_n) : n \in \mathbb{N}_{>0}\right\} = 1$$

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