Calculating bounds on expected return and first passage times in finite-state imprecise birth-death chains

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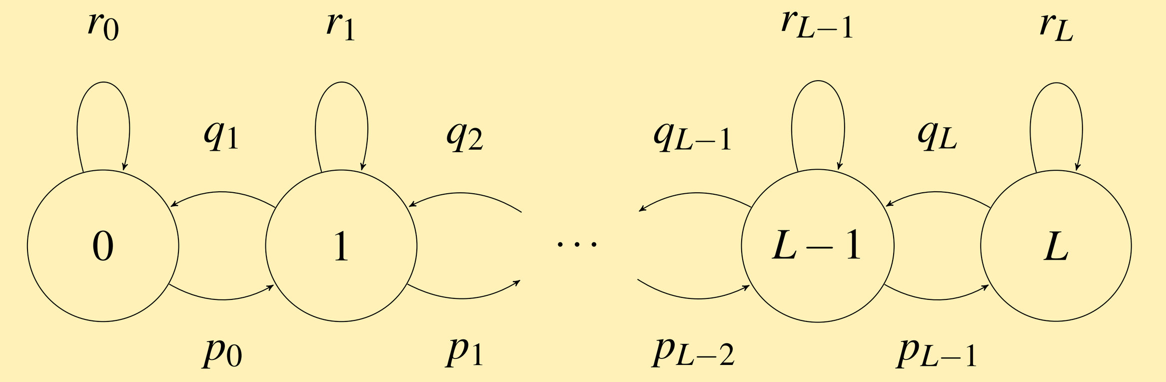
Outline

We provide simple methods for computing **exact bounds on expected return and first passage times in finite-state birth-death chains**, when the transition probabilities are **imprecise**, in the sense that they are only known to belong to convex closed sets of probability mass functions. We describe a precise birth-death chain and then we define an imprecise version of it. We present the main results regarding expected lower first passage and return times. We also discuss the special case where the local models are **linear-vacuous mixtures** and finally we show some numerical results through a general example.

(Imprecise) Birth-Death Chains

Precise Case Consider a finite state space $\mathscr{X} = \{0,\dots,L\}$, with $L \in \mathbb{N}$. At any time point $n \in \mathbb{N}$, the state of the chain is represented by a random variable X_n taking values in $\mathscr{X}$. The sequence $X_1,\dots,X_n$ is denoted by $X_{1:n}$ and takes values $x_{1:n} := x_1,\dots,x_n$ in $\mathscr{X}^n$. A birth-death chain is characterised by a stochastic and tridiagonal matrix P.

$$P = \begin{pmatrix} r_0 & p_0 & 0 & \cdots & \cdots & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & q_{L-1} & r_{L-1} & p_{L-1} \\ 0 & \cdots & \cdots & 0 & q_L & r_L \end{pmatrix} \quad \Rightarrow$$



A birth-death chain is a Markov chain and therefore satisfies the **Markov condition**, which has as follows

$$E_{n+1}(\cdot|x_{1:n}) = E_{n+1}(\cdot|x_n) \text{ for all } x_{1:n} \in \mathscr{X}^n.$$

Imprecise Case Consider a matrix P, whose probability mass functions (p.m.f.) are not precisely known. For every $i \in \mathscr{X}$, the p.m.f. of the i row is only known to belong to some credal set $\mathscr{M}_i$, whose elements are of the following form:

$$\phi_i(j) = \begin{cases} q_i & \text{if } j = i-1 \\ r_i & \text{if } j = i \\ p_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \quad i \in \mathscr{X}\setminus\{0,L\} \qquad \phi_0(j) = \begin{cases} r_0 & \text{if } j = 0 \\ p_0 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_L(j) = \begin{cases} q_L & \text{if } j = L-1 \\ r_L & \text{if } j = L \\ 0 & \text{otherwise} \end{cases}$$

where r_0, p_0, r_L, q_L and q_i, r_i, p_i for all $i \in \mathscr{X}\setminus\{0,L\}$ are strictly positive **(POS)**. The set $\mathscr{M}_i$ is a subset of the simplex $\Sigma_{\mathscr{X}}$ of all p.m.f. on the state space. For any $i \in \mathscr{X}$, the subset of $\Sigma_{\mathscr{X}}$ containing p.m.f. of the form ϕ_i will be denoted by $\Sigma_{\mathscr{X}_i}$. The **imprecise Markov condition** is defined as

$$\underline{E}_{n+1}(\cdot|x_{1:n}) = \underline{E}_{n+1}(\cdot|x_n) := \underline{E}(\cdot|x_n) \textbf{ (M)} \text{ and similarly for the upper case.}$$

From **(M)**, we derive—based on the notion of **submartingales**—global uncertainty models. Our models satisfy a version of the law of iterated expectation. That is, for every $n \in \mathbb{N}$ and every extended real-valued function g on $\mathscr{X}^{\mathbb{N}}$, it holds that $\underline{E}_{n+1:\infty}(g(X_{n+1:\infty})|i) = \underline{E}_{n+2:\infty}(g(X_{n+2:\infty})|i)$ **(IN)** and by defining $f'(i') := \underline{E}_{n+2:\infty}(g(i', X_{n+2:\infty})|i')$ for all $i' \in \mathscr{X}$, we have that $\underline{E}_{n+1:\infty}(g(X_{n+1:\infty})|i) = \underline{E}_{n+1}(f'|i) = \underline{E}(f'|i)$ **(GLM)**

First passage times

The first passage time from i to j with $i, j \in \mathscr{X}$ is

$$\tau_{i\to j}(i, X_{n+1:\infty}) := \begin{cases} 1 & X_{n+1} = j \\ 1 + \tau_{X_{n+1}\to j}(X_{n+1}, X_{n+2:\infty}) & X_{n+1} \neq j \end{cases}$$
$$= 1 + \mathbb{I}_{j^c}(X_{n+1})\tau_{X_{n+1}\to j}(X_{n+1}, X_{n+2:\infty}) \textbf{ (FPT)}$$

where $\mathbb{I}_{j^c}$ is the indicator of $j^c := \mathscr{X}\setminus\{j\}$.

Due to **(IN)**, the lower and upper expected values are

$$\underline{\tau}_{i\to j,n} := \underline{E}_{n+1:\infty}(\tau_{i\to j}(i, X_{n+1:\infty})|i)$$
$$\overline{\tau}_{i\to j,n} := \overline{E}_{n+1:\infty}(\tau_{i\to j}(i, X_{n+1:\infty})|i)$$

and they are independent of n.

By combining **(FPT)** with **(GLM)** and **(POS)** we have nice expressions for $\underline{\tau}_{0\to j}$, $\underline{\tau}_{L\to j}$ and $\underline{\tau}_{i\to j}$, for all $i \in \mathscr{X}\setminus\{0,L\}$ and also for the respective upper ones. All our results are presented with respect to expected lower values and the upper ones follow analogously.

We distinguish between expected lower **upward** and **downward** first passage times. Combining these two, we calculate **return** times.

Upward (U) We first calculate $\underline{\tau}_{i\to j}$ when $i < j$.

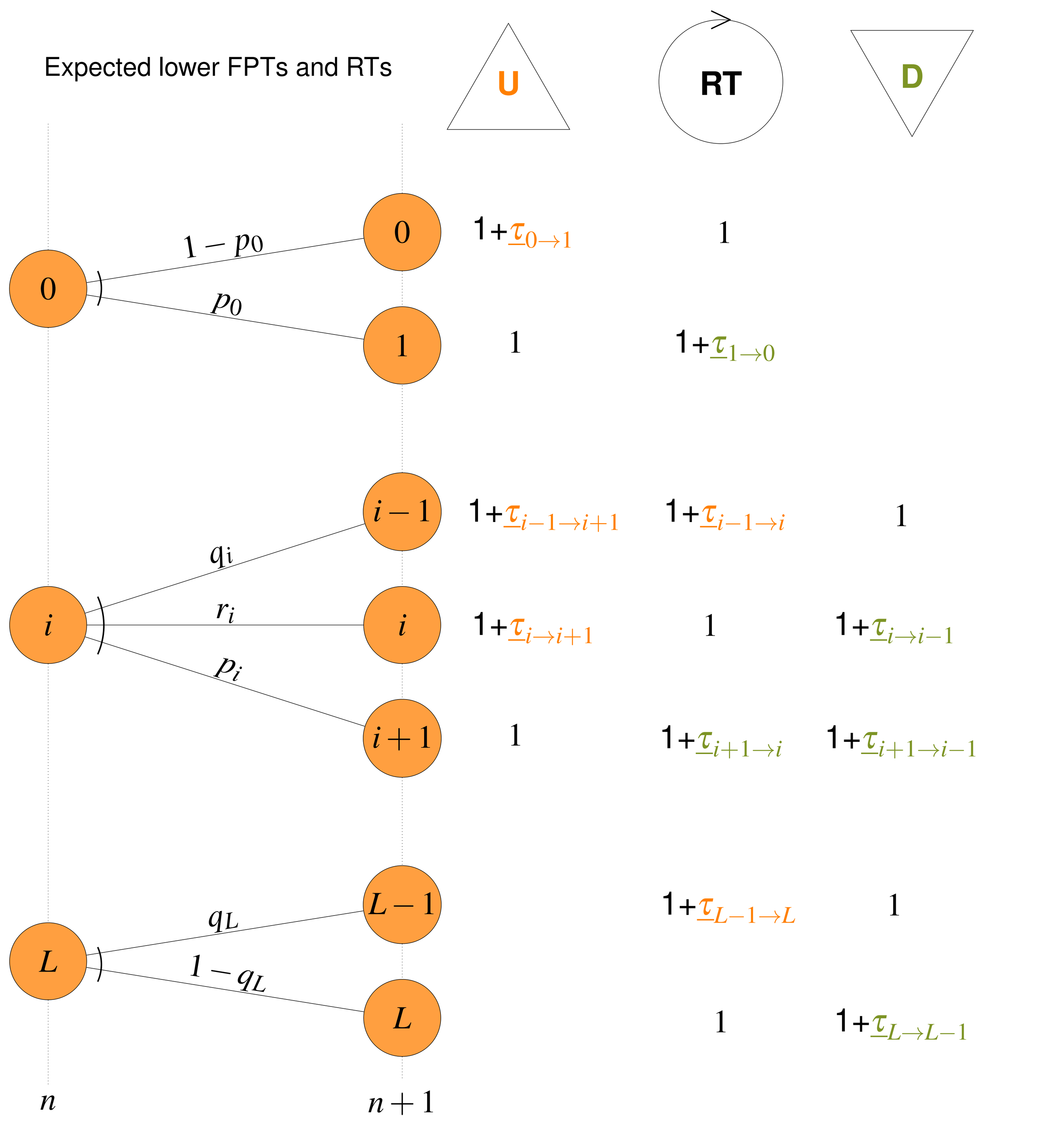
$$\underline{\tau}_{0\to 1} = \frac{1}{\underline{p}_0}.$$

We can prove that for all $i \in \mathscr{X}\setminus\{0,L\}$

$$\min_{\phi_i\in\mathscr{M}_i}\{q_i\underline{\tau}_{i-1\to i} - p_i\underline{\tau}_{i\to i+1}\} = -1,$$

and that it is a strictly decreasing function on $\underline{\tau}_{i\to i+1}$. Using this, we can calculate $\underline{\tau}_{i\to i+1}$ recursively. The expected value $\underline{\tau}_{i\to i+1}$ can be calculated by means of a bisection method as long as $\underline{\tau}_{i-1\to i}$ is calculated, which can be calculated as long as $\underline{\tau}_{i-2\to i-1}$ is calculated and so on. Finally, for all $i \in \mathscr{X}$, with $i < j$, we have that

$$\underline{\tau}_{i\to j} = \sum_{k=i}^{j-1}\underline{\tau}_{k\to k+1}$$



First passage times

Downward (D) Similarly to the upward case, we calculate $\underline{\tau}_{i\to j}$, when $i > j$.

$$\underline{\tau}_{L\to L-1} = \frac{1}{\underline{q}_L}$$

For all $i \in \mathscr{X}\setminus\{0,L\}$, we have that

$$\min_{\phi_i\in\mathscr{M}_i}\{-q_i\underline{\tau}_{i\to i-1} + p_i\underline{\tau}_{i+1\to i}\} = -1$$

and for all $i \in \mathscr{X}$, with $i > j$, we have that

$$\underline{\tau}_{i\to j} = \sum_{k=j}^{i-1}\underline{\tau}_{k+1\to k}$$

Return Times (RT)

$$\underline{\tau}_{0\to 0} = 1 + \min_{\phi_0\in\mathscr{M}_0}\{p_0\underline{\tau}_{1\to 0}\} = 1 + \underline{p}_0\underline{\tau}_{1\to 0},$$
$$\underline{\tau}_{L\to L} = 1 + \min_{\phi_L\in\mathscr{M}_L}\{q_L\underline{\tau}_{L-1\to L}\} = 1 + \underline{q}_L\underline{\tau}_{L-1\to L}$$

and, for all $i \in \mathscr{X}\setminus\{0,L\}$:

$$\underline{\tau}_{i\to i} = 1 + \min_{\phi_i\in\mathscr{M}_i}\{q_i\underline{\tau}_{i-1\to i} + p_i\underline{\tau}_{i+1\to i}\}$$

Linear-vacuous mixtures

Given the following p.m.f.

$$\phi_0^* = (r_0^*, p_0^*, 0, \dots, 0),\ \phi_L^* = (0, \dots, 0, q_L^*, r_L^*)$$

and, for all $i \in \mathscr{X}\setminus\{0,L\}$

$$\phi_i^* = (0, \dots, 0, q_i^*, r_i^*, p_i^*, 0, \dots, 0)$$

where $r_0^*, p_0^*, r_L^*, q_L^*$ and q_i^*, r_i^*, p_i^*, for all $i \in \mathscr{X}\setminus\{0,L\}$, are strictly positive. Given also a real-valued $\varepsilon_i \in [0,1)$, for all $i \in \mathscr{X}$, we define the linear-vacuous local credal sets:

$$\mathscr{M}_0 = \left\{(1-\varepsilon_0)\phi_0^* + \varepsilon_0\phi_0' : \phi_0' \in \Sigma_{\mathscr{X}_0}\right\},$$
$$\mathscr{M}_L = \left\{(1-\varepsilon_L)\phi_L^* + \varepsilon_L\phi_L' : \phi_L' \in \Sigma_{\mathscr{X}_L}\right\}$$

and, for all $i \in \mathscr{X}\setminus\{0,L\}$,

$$\mathscr{M}_i = \left\{(1-\varepsilon_i)\phi_i^* + \varepsilon_i\phi_i' : \phi_i' \in \Sigma_{\mathscr{X}_i}\right\},$$

Furthermore, for all $i \in \mathscr{X}\setminus\{0\}$, we define

$$\underline{q}_i := (1-\varepsilon_i)q_i^* \text{ and } \overline{q}_i := (1-\varepsilon_i)q_i^* + \varepsilon_i$$

and, for all $i \in \mathscr{X}\setminus\{L\}$,

$$\underline{p}_i := (1-\varepsilon_i)p_i^* \text{ and } \overline{p}_i := (1-\varepsilon_i)p_i^* + \varepsilon_i,$$

For the expected lower upward and downward first passage times, we have that

$$\underline{\tau}_{i\to i+1} = \frac{1}{\underline{p}_i} + \frac{\overline{q}_i}{\underline{p}_i}\underline{\tau}_{i-1\to i} = \sum_{k=0}^{i}\frac{\prod_{\ell=k+1}^{i}\overline{q}_\ell}{\prod_{m=k}^{i}\underline{p}_m}.$$

$$\underline{\tau}_{i\to i-1} = \frac{1}{\underline{q}_i} + \frac{\overline{p}_i}{\underline{q}_i}\underline{\tau}_{i+1\to i} = \sum_{k=i}^{L}\frac{\prod_{\ell=i}^{k-1}\overline{p}_\ell}{\prod_{m=i}^{k}\underline{q}_m}$$

and for the expected lower return times that

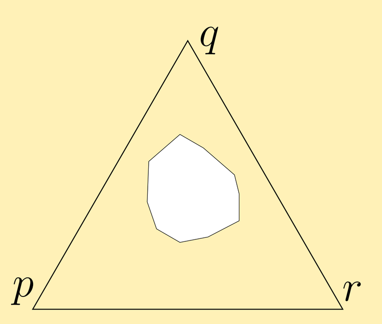
$$\underline{\tau}_{i\to i} = 1 + \underline{q}_i\underline{\tau}_{i-1\to i} + \underline{p}_i\underline{\tau}_{i+1\to i}$$

General example

Consider an imprecise birth-death chain with $\mathscr{X} = \{0,1,2,3,4\}$. Let $\mathscr{M}_0$ be determined by $\underline{p}_0 = 0.15$ and $\overline{p}_0 = 0.4$ and $\mathscr{M}_L$ by $\underline{q}_L = 0.2$ and $\overline{q}_L = 0.6$. For $i \in \mathscr{X}\setminus\{0,L\}$, the extreme points of the credal set $\mathscr{M}_i$ are characterised by the following ten triplets of the form (q_i, r_i, p_i):

$$(0.65, 0.15, 0.2), (0.6, 0.25, 0.15),$$
$$(0.5, 0.4, 0.1), (0.43, 0.45, 0.12),$$
$$(0.33, 0.5, 0.17), (0.27, 0.43, 0.3),$$
$$(0.25, 0.35, 0.4), (0.3, 0.25, 0.45),$$
$$(0.4, 0.17, 0.43), (0.55, 0.1, 0.35)$$

$\Rightarrow$



$\underline{\tau}_{0\to 1}$	2.5	$\underline{\tau}_{4\to 3}$	1.666
$\underline{\tau}_{1\to 2}$	3.889	$\underline{\tau}_{3\to 2}$	2.051
$\underline{\tau}_{2\to 3}$	4.814	$\underline{\tau}_{2\to 1}$	2.169
$\underline{\tau}_{3\to 4}$	5.432	$\underline{\tau}_{1\to 0}$	2.206
$\overline{\tau}_{0\to 1}$	6.666	$\overline{\tau}_{4\to 3}$	5
$\overline{\tau}_{1\to 2}$	43.333	$\overline{\tau}_{3\to 2}$	12
$\overline{\tau}_{2\to 3}$	226.666	$\overline{\tau}_{2\to 1}$	23.2
$\overline{\tau}_{3\to 4}$	1143.333	$\overline{\tau}_{1\to 0}$	41.12