Decisions under risk and partial knowledge modelling uncertainty and risk aversion

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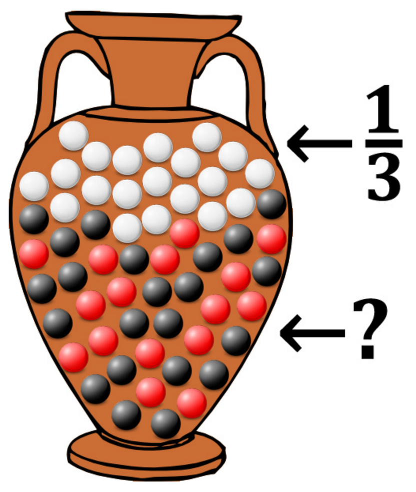
Lotteries and imprecise probabilities

Ellsberg's paradox



	w	b	r
L_1	100€	0€	0€
L_2	0€	0€	100€
L_3	0€	100€	100€
L_4	100€	100€	0€

$$\mathcal{P} = \left\{P^\theta : P^\theta(\{w\}) = \tfrac{1}{3}, P^\theta(\{b\}) = \theta, P^\theta(\{r\}) = \tfrac{2}{3} - \theta, \theta \in \left[0, \tfrac{2}{3}\right]\right\}$$

There is no $P^\theta \in \mathcal{P}$ s.t. there exists $u : \{0, 100\} \to \mathbb{R}$ whose EU on L_i's represents the preferences

$$L_2 \prec L_1,\ L_4 \prec L_3.$$

Generalized convex lotteries

Consider a **gc-lottery** $L = (\wp(X), \varphi_L) = (\wp(X), m_L)$ where:

- $X = \{x_1, \ldots, x_n\}$, finite set of **prizes**;
- $\varphi_L : \wp(X) \to [0, 1]$, **convex capacity** with **Möbius inversion** $m_L : \wp(X) \to \mathbb{R}$

Convex combination: Given $L_1, \ldots, L_t$ gc-lotteries and $\mathbf{k} = (k_1, \ldots, k_t)$ with $k_i \geq 0$ for $i = 1, \ldots, t$ and $\sum_{i=1}^t k_i = 1$ define

$$\mathbf{k}(L_1, \ldots, L_t) = \left(\begin{array}{c} A \\ \sum_{i=1}^t k_i m_{L_i}(A) \end{array}\right)_{A \in \wp(X)\setminus\{\emptyset\}}.$$

Degenerate gc-lotteries: For $A \in \wp(X)\setminus\{\emptyset\}$, δ_A is the gc-lottery such that $m_{\delta_A}(A) = 1$.

Preferences on gc-lotteries

- $\leq^*$, total preorder on X with asymmetric and symmetric parts $<^*$ and $=^*$
- $\mathcal{L}$, a set of gc-lotteries on X
- $(\precsim, \prec)$, **strengthened preference relation** on $\mathcal{L}$

Assumption (A0)

(A0) $\mathcal{L}_0 = \{\delta_{\{x\}} : x \in X\} \subseteq \mathcal{L}$ and $x \leq^* x'$ if and only if $\delta_{\{x\}} \precsim \delta_{\{x'\}}$, for $x, x' \in X$.

Aggregated Möbius inversion: Defined on $X^* = X_{/=^*} = \{[x_{i_1}], \ldots, [x_{i_m}]\}$, with $[x_{i_1}] <^* \ldots <^* [x_{i_m}]$ and $E_i = \{x_i, \ldots, x_n\}$ for $i = 1, \ldots, n$, as

$$M_L([x_{i_j}]) = \sum_{x_i \in [x_{i_j}]} \sum_{\{x_i\} \subseteq B \subseteq E_i} m_L(B).$$

Definition [Choquet rationality]

(gc-CR) For all $h \in \mathbb{N}$ and $L_i, L_i' \in \mathcal{L}$ with $L_i \precsim L_i'$ $(i = 1, \ldots, h)$, if

$$\mathbf{k}(M_{L_1}, \ldots, M_{L_h}) = \mathbf{k}(M_{L_1'}, \ldots, M_{L_h'})$$

with $\mathbf{k} = (k_1, \ldots, k_h)$, $k_i > 0$ $(i = 1, \ldots, h)$ and $\sum_{i=1}^h k_i = 1$, then it cannot be $L_i \prec L_i'$ for any $i = 1, \ldots, h$.

CEU representability

CEU representation of $(\precsim, \prec)$ **on** $\mathcal{L}$:

$$\begin{cases} L \precsim L' \Longrightarrow \mathrm{CEU}(L) \leq \mathrm{CEU}(L'), \\ L \prec L' \Longrightarrow \mathrm{CEU}(L) < \mathrm{CEU}(L'). \end{cases} \quad \text{where } \mathrm{CEU}(L) = \oint u \, d\varphi_L = \sum_{[x_{i_j}] \in X^*} u(x_{i_j}) M_L([x_{i_j}])$$

Theorem [CEU representability]

Let $\mathcal{L}$ be a finite set of g-lotteries on $X = \{x_1, \ldots, x_n\}$ and let $\leq^*$ be a total preorder on X. For a strengthened preference relation $(\precsim, \prec)$ on $\mathcal{L}$ satisfying **(A0)** the following statements are equivalent.

(i) $(\precsim, \prec)$ is Choquet rational (i.e., it satisfies **(gc-CR)**);

(ii) there exists an increasing function $u : X \to \mathbb{R}$, whose CEU functional on $\mathcal{L}$ represents $(\precsim, \prec)$.

The particular case of money payoffs

Suppose $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}$ and $\leq^*$ coincides with $\leq$ on $\mathbb{R}$ with $x_1 < \ldots < x_n$

Let $\mathbf{k}_i = (k_i, 1 - k_i)$ with $k_i = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}$, for $i = 2, \ldots, n-1$, and define:

$$\mathcal{L}_1 = \left\{\mathbf{k}_i\left(\delta_{\{x_{i-1}\}}, \delta_{\{x_{i+1}\}}\right) : i = 2, \ldots, n-1\right\}$$

Assumptions (A1) and (A1*)

(A1) $\mathcal{L}_1 \subseteq \mathcal{L}$ and $\mathbf{k}_i\left(\delta_{\{x_{i-1}\}}, \delta_{\{x_{i+1}\}}\right) \prec \delta_{\{x_i\}}$ or $\mathbf{k}_i\left(\delta_{\{x_{i-1}\}}, \delta_{\{x_{i+1}\}}\right) \sim \delta_{\{x_i\}}$.

(A1*) $\mathcal{L}_1 \subseteq \mathcal{L}$ and $\mathbf{k}_i\left(\delta_{\{x_{i-1}\}}, \delta_{\{x_{i+1}\}}\right) \prec \delta_{\{x_i\}}$.

Proposition [Risk aversion in case of money payoffs]

Assume $(\precsim, \prec)$ satisfies **(A0)** and **(gc-CR)** and let u be a utility whose CEU represents $(\precsim, \prec)$. The following statements hold:

(i) if **(A1)** holds then u extends to an increasing concave function $v \in C^0([x_1, x_n])$;

(ii) if **(A1*)** holds then u extends to an increasing strictly concave function $w \in C^2([x_1, x_n])$.

Cumulative aggregated Möbius inversion: For every $x \in \mathbb{R}$ and $L \in \mathcal{L}$ define

$$F_L(x) = \sum_{x_i \leq x} M_L(x_i)$$

Proposition [S.o.s.d. in case of money payoffs]

Assume **(A0)** and **(A1)** are satisfied. If $(\precsim, \prec)$ satisfies **(gc-CR)** then for every complete preference relation $\precsim'$ on $\mathcal{L}$ extending $(\precsim, \prec)$ and satisfying **(gc-CR)** the following condition holds for every $L_1, L_2 \in \mathcal{L}$:

(S2) if $\int_{-\infty}^x F_{L_1}(t)\,dt \leq \int_{-\infty}^x F_{L_2}(t)\,dt$ for every $x \in \mathbb{R}$, it cannot be $L_1 \prec' L_2$.

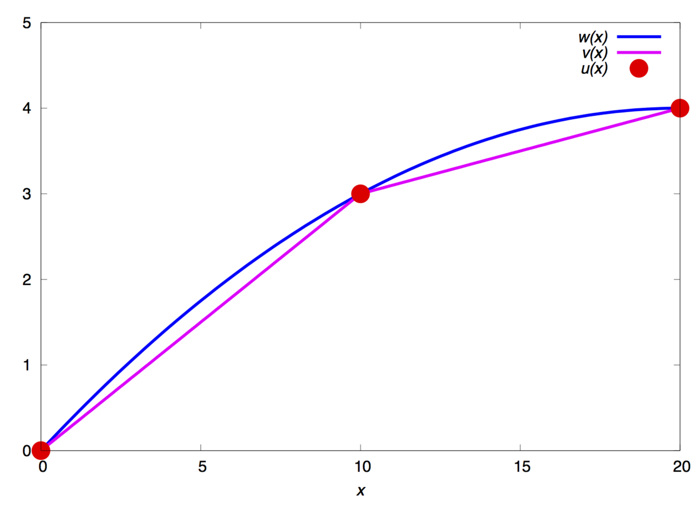
An example for $X = \{0, 10, 20\}$

$(\precsim, \prec)$ satisfying **(A0)**:

$L_2 \prec L_1, L_3 \prec L_1$

	$\{0\}$	$\{10\}$	$\{20\}$	$\{0,10\}$	$\{0,20\}$	$\{10,20\}$	X
L_1	0.4	0.1	0.2	0.1	0.1	0.2	−0.1
L_2	0.5	0.5	0	0	0	0	0
L_3	0.2	0	0.2	0	0.6	0	0

X	0	10	20
M_{L_1}	0.5	0.3	0.2
M_{L_2}	0.5	0.5	0
M_{L_3}	0.8	0	0.2



Extension of a Choquet rational relation

Theorem [Extension]

Let $X = \{x_1, \ldots, x_n\}$ be a finite set with a total prorder $\leq^*$, $\mathcal{L}$ and $\mathcal{L}'$ finite sets of gc-lotteries on X, with $\mathcal{L} \subseteq \mathcal{L}'$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$ satisfying **(A0)**. Then if $(\precsim, \prec)$ satisfies condition **(gc-CR)** there exists a family $\{\precsim^\gamma : \gamma \in \Gamma\}$ of complete relations on $\mathcal{L}'$ satisfying **(gc-CR)** which extend $(\precsim, \prec)$. Moreover, denoting with $\prec^\gamma$ and $\sim^\gamma$, respectively, the strict and symmetric parts of $\precsim^\gamma$, for $\gamma \in \Gamma$, condition **(gc-CR)** singles out the relations

$$\prec^* = \bigcap\{\prec^\gamma : \gamma \in \Gamma\} \quad \text{and} \quad \sim^* = \bigcap\{\sim^\gamma : \gamma \in \Gamma\}.$$

The extension of $(\precsim, \prec)$ on a new pair of gc-lotteries (F, G) can be computed solving at most three linear systems

$$\mathcal{S}^{\prec^*} : \begin{cases} A'w > 0 \\ Bw \geq 0 \end{cases} \qquad \mathcal{S}^{\succ^*} : \begin{cases} A''w > 0 \\ Bw \geq 0 \end{cases} \qquad \mathcal{S}^{\sim^*} : \begin{cases} Aw > 0 \\ B'w \geq 0 \end{cases}$$

function Extension($(\precsim, \prec)$, F, G)
  if $\mathcal{S}^{\prec^*}$ and $\mathcal{S}^{\succ^*}$ are solvable **then** free preference between F and G
  else if $\mathcal{S}^{\prec^*}$ is solvable and $\mathcal{S}^{\sim^*}$ is not **then** it must be $F \prec G$
  else if $\mathcal{S}^{\succ^*}$ is solvable and $\mathcal{S}^{\sim^*}$ is not **then** it must be $G \prec F$
  else if $\mathcal{S}^{\prec^*}$ and $\mathcal{S}^{\sim^*}$ are solvable **then** it cannot be $G \prec F$
  else if $\mathcal{S}^{\succ^*}$ and $\mathcal{S}^{\sim^*}$ are solvable **then** it cannot be $F \prec G$
  else it must be $F \sim G$
end function