Comonotone lower probabilities for bivariate and discrete structures

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Abstract

Two random variables are comonotone when there is an increasing relation between them, in the sense that they increase or decrease simultaneously. This notion has been widely investigated in probability theory, and is related to the theory of copulas. This contribution defines and characterizes the notion of comonotonicity for lower probabilities. Also, we provide some sufficient conditions allowing to define a comonotone belief function with fixed marginals and characterize comonotone bivariate p-boxes.

Some notation

- We deal with finite and ordered spaces $\mathcal{X} = \{x_1, \ldots, x_n\}$, $\mathcal{Y} = \{y_1, \ldots, y_m\}$ and $\mathcal{X} \times \mathcal{Y}$.
- We consider an imprecisely defined probability $P_{X,Y}$ defined on $\mathcal{X} \times \mathcal{Y}$ and we model it by means of a coherent lower probability $\underline{P}$.
- $F_{X,Y}$ denotes the bivariate cdf associated with $P_{X,Y}$ and by F_X, F_Y its marginal cdfs.
- We denote by $(\underline{F}, \overline{F})$ the bivariate p-box associated with $\underline{P}$ and by $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ the marginal p-boxes.

Comonotonicity for $P_{X,Y}$

Comonotonicity means that there exists an increasing relation between both components.
A precise probability $P_{X,Y}$ is called comonotone when one of the following equivalent conditions hold:

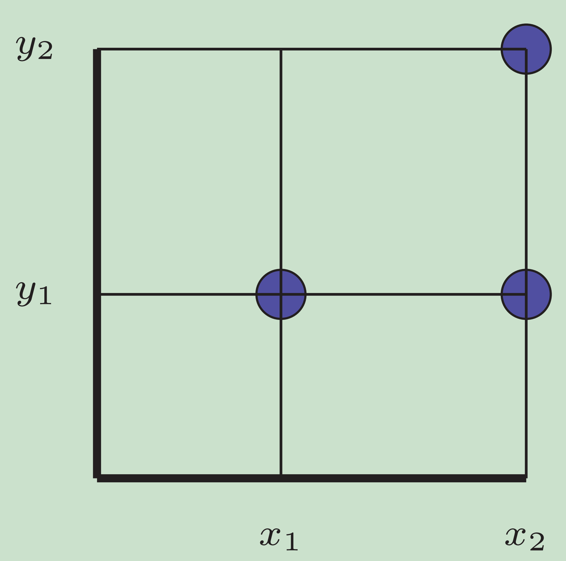
- $\text{supp}(P_{X,Y})$ is an increasing set on $\mathcal{X} \times \mathcal{Y}$.
- $F_{X,Y}(x,y) = \min\{F_X(x), F_Y(y)\}$ for any $(x,y) \in \mathcal{X} \times \mathcal{Y}$.
- For any $(x,y) \in \mathcal{X} \times \mathcal{Y}$, either

$$P(\{(u,v) \mid u \le x, v > y\}) = 0 \text{ or}$$
$$P(\{(u,v) \mid u > x, v \le y\}) = 0.$$

Comonotonicity for $\underline{P}$

Definition 1 *A lower probability $\underline{P}$ on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is comonotone when any $P \in \mathcal{M}(\underline{P})$ is comonotone.*

Example 2 *Let $\mathcal{X} \times \mathcal{Y} = \{x_1, x_2\} \times \{y_1, y_2\}$ and consider $\underline{P}$ such that $\overline{P}(\{(x_1, y_2)\}) = 0$. Then, the support of any probability $P \in \mathcal{M}(\underline{P})$ is included in the blue zone $\{(x_1,y_1),(x_2,y_1),(x_2,y_2)\}$:*



Then, any $P \in \mathcal{M}(\underline{P})$ is comonotone, and therefore also is $\underline{P}$.

Characterizing comonotone coherent lower probabilities

Theorem 3 *A coherent $\underline{P}$ on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is comonotone iff $\forall (x,y) \in \mathcal{X} \times \mathcal{Y}$ either:*

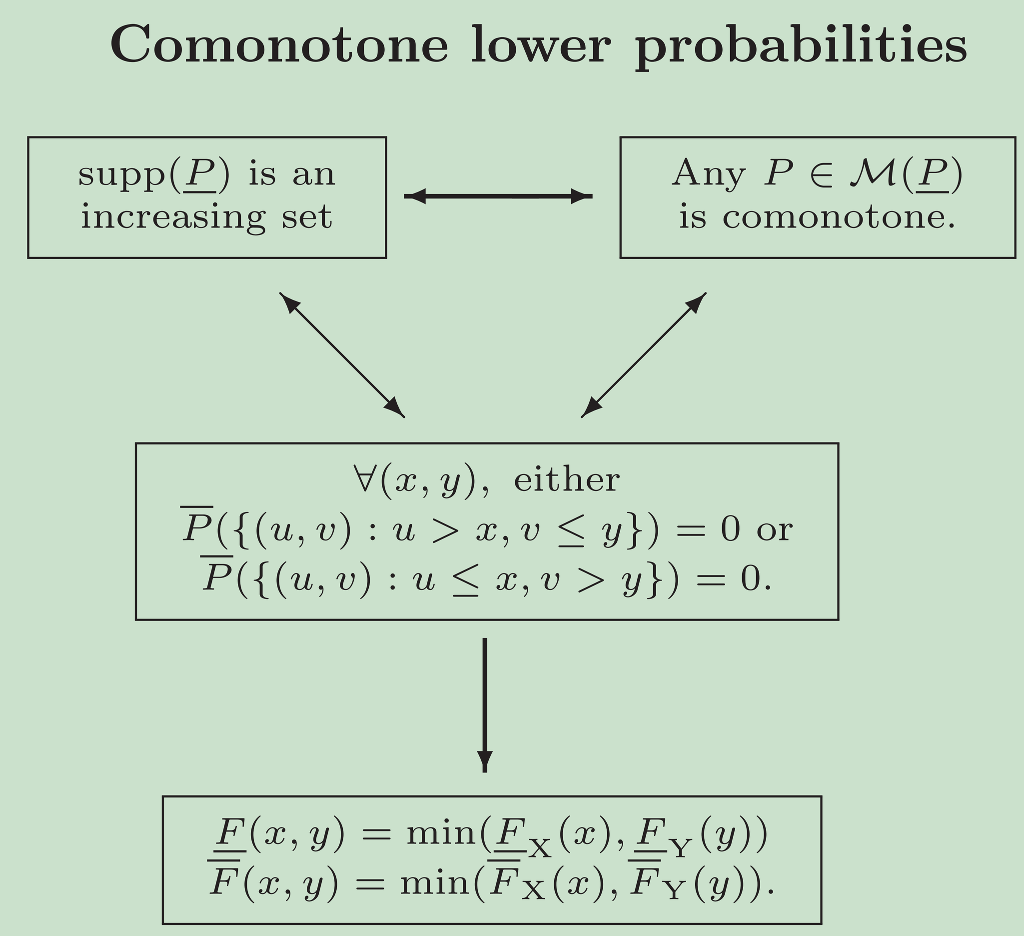
$$\overline{P}(\{(u,v) \mid u \le x, v > y\}) = 0 \text{ or}$$
$$\overline{P}(\{(u,v) \mid u > x, v \le y\}) = 0.$$

Theorem 4 *A coherent $\underline{P}$ is comonotone iff the support of $\underline{P}$, $\text{supp}\underline{P} = \cup_{P \in \mathcal{M}(\underline{P})} \text{supp}(P)$ is an increasing set on $\mathcal{X} \times \mathcal{Y}$.*

Theorem 5 *If a coherent $\underline{P}$ on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is comonotone, then $\forall (x,y) \in \mathcal{X} \times \mathcal{Y}$:*

$$\underline{F}(x,y) = \min\{\underline{F}_X(x), \underline{F}_Y(y)\}.$$
$$\overline{F}(x,y) = \min\{\overline{F}_X(x), \overline{F}_Y(y)\}.$$

However, the converse does not hold in general.



Building comonotone belief functions with fixed marginals

Precise probabilities: marginals P_X and P_Y define a unique joint comonotone probability $P_{X,Y}$.

Lower probabilities: not any marginals $\underline{P}_X, \underline{P}_Y$ define a comonotone $\underline{P}$. Even when they do, the comonotone $\underline{P}$ may not be unique.

Important question: can we give sufficient conditions on the marginals to assure the existence of a comonotone $\underline{P}$?

Proposition 6 *Given two marginal belief functions Bel_X and Bel_Y there exists a comonotone belief function with these marginals when Bel_X and Bel_Y satisfy one of the following sufficient conditions:*

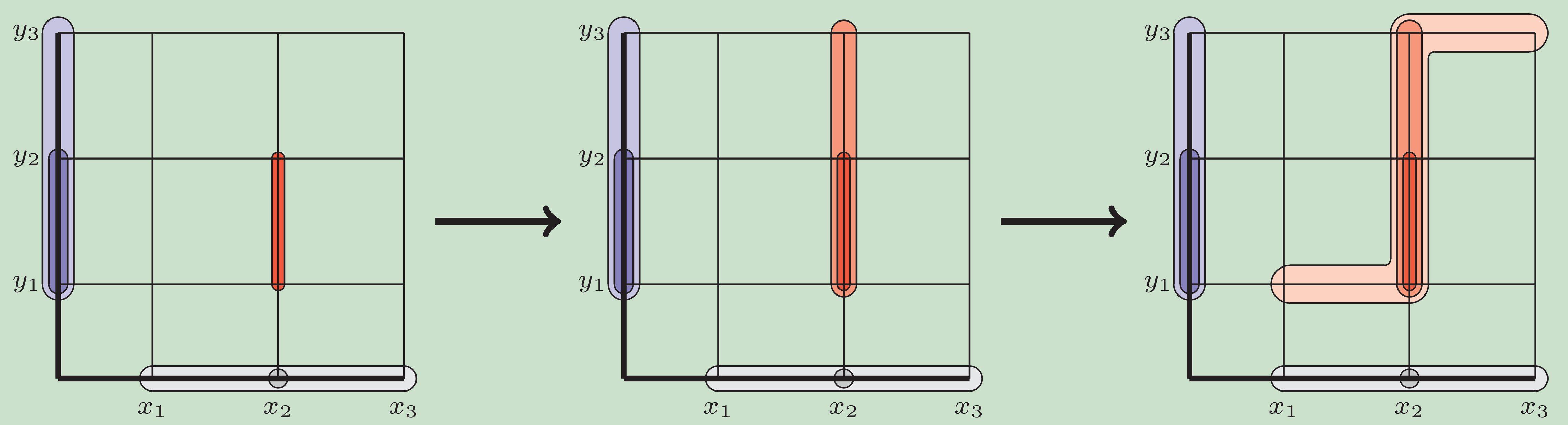
1. *The focal sets of Bel_X and Bel_Y are nested.*
2. *The focal sets of Bel_X and Bel_Y are ordered.*

Example 7 *Consider $\mathcal{X} = \{x_1, x_2, x_3\}$, $\mathcal{Y} = \{y_1, y_2, y_3\}$ and Bel_X, Bel_Y given by:*

$$m_X(\{x_2\}) = 0.6, \quad m_X(\{x_1,x_2,x_3\}) = 0.4, \quad m_Y(\{y_1,y_2\}) = 0.2, \quad m_Y(\{y_1,y_2,y_3\}) = 0.8.$$

We split the focal sets in of both belief functions order to obtain the same mass distribution:

$$m_X(\{x_2\}) = 0.2, \quad m_X(\{x_2\}) = 0.4, \quad m_X(\{x_1,x_2,x_3\}) = 0.4.$$
$$m_Y(\{y_1,y_2\}) = 0.2, \quad m_Y(\{y_1,y_2,y_3\}) = 0.4 \quad m_Y(\{y_1,y_2,y_3\}) = 0.4.$$



Comonotone bivariate p-boxes

- We can use a bivariate p-box $(\underline{F}, \overline{F})$ to model the imprecise information about $F_{X,Y}$. $(\underline{F}, \overline{F})$ is comonotone when $\underline{P}_{(\underline{F}, \overline{F})}$ is comonotone.
- However, bivariate p-boxes are not very adequate to model comonotonicity because they must satisfy very restrictive conditions.
- In fact, one of the marginals must be precise and $\underline{P}_{(\underline{F}, \overline{F})}$ must be a belief function.

Example 8

y_5	[0'4, 0'4]	[0'7, 0'7]	*[1,1]*
y_4	[0'4, 0'4]	[0'7, 0'7]	*[0'8,1]*
y_3	[0'4, 0'4]	*[0'7,0'7]*	*[0'8,1]*
y_2	*[0'4,0'4]*	*[0'5,0'5]*	[0'5, 0'5]
y_1	*[0'1,0'1]*	[0'1, 0'1]	[0'1, 0'1]
$(\underline{F}, \overline{F})$	x_1	x_2	x_3

Conclusions

- This work investigates the notion of comonotonicity for lower probabilities.
- We have characterized comonotonicity and established a connection with Sklar's Theorem.
- For marginal belief functions, there are situations in which it is possible to build a comonotone belief function with the fixed marginals.
- Bivariate p-boxes are not adequate to model comonotonicity.