Eliciting Sets of Acceptable Gambles

The CWI World Cup Competition

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Introduction

This poster introduces a procedure for eliciting coherent sets of acceptable gambles on three-outcome possibility spaces. We also discuss a real-life experiment conducted as an exploratory test of this elicitation interface; it was organized around the 2014 FIFA World Cup.

> Because I'm inside a yellow box, I'm a running example or some other illustration!

Essential Concepts

Possibility space Ω, finite set of possible experimental outcomes.

> $\Omega = \{W, D, L\}$, for 'Win', 'Draw', and 'Loss'.

Gamble A real-valued function g on Ω, representing a positive or negative payoff $g(\omega)$, with $\omega \in \Omega$ a possible outcome.

> An example: $g = 4I_L - I_W = (-1, 0, 4)$, with I indicator function notation.

Acceptable gamble An elicitee finds a gamble g acceptable if she is committed to receiving the payoff $g(\omega)$ once the actual outcome $\omega \in \Omega$ is determined.

Assessment $\mathcal{A}$, a finite set of gambles assessed to be acceptable.

> $\mathcal{A} = \{6I_W - 1, 3I_{WD} - 1, 4I_L - I_W\}$.

Coherence axioms A coherent set of acceptable gambles $\mathcal{D}$ should satisfy:

Avoiding Sure Loss: $g < 0 \Rightarrow g \notin \mathcal{D}$,
Addition: $g, h \in \mathcal{D} \Rightarrow g + h \in \mathcal{D}$,
Positive Homogeneity: $g \in \mathcal{D}, \lambda_g > 0 \Rightarrow \lambda_g g \in \mathcal{D}$,
Accepting Partial Gains: $g \geq 0 \Rightarrow g \in \mathcal{D}$.

$\mathcal{D}$ is a convex cone that includes the positive orthant and does not intersect the negative one.

Natural extension The smallest set of acceptable gambles that includes an assessment $\mathcal{A}$,

$$\mathcal{D} = \left\{ f + \sum_{g \in \mathcal{A}} \lambda_g g : f \geq 0, \lambda_g \geq 0 \right\}.$$

> Intersection of $\mathcal{D}$ with the plane of gambles whose payoffs sum to one:
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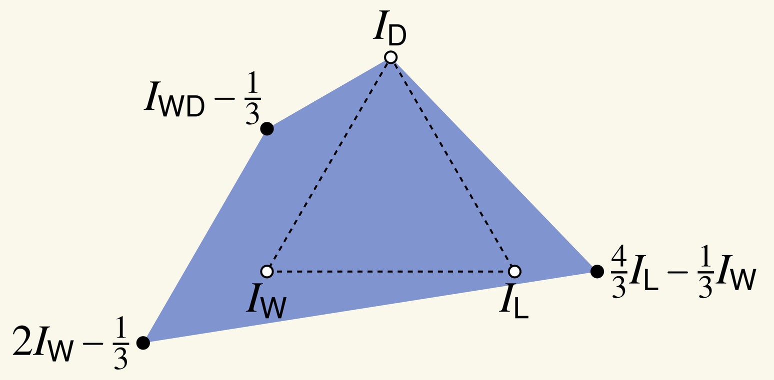
> 
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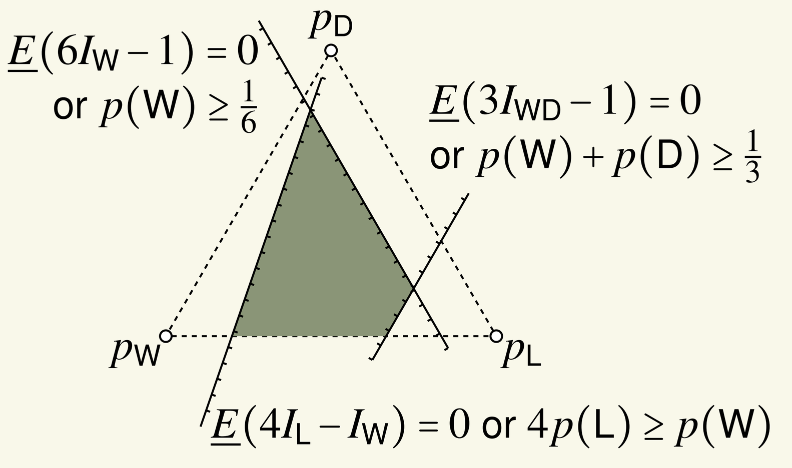
>
> (Dashed triangle delimits positive octant.)

Lower expectation or prevision The supremum acceptable buying price for the gamble h,

$$\underline{E}(h) := \sup\{\alpha \in \mathbb{R} : h - \alpha \in \mathcal{D}\}.$$

Credal set A convex subset of the probability simplex,

$$\mathcal{M} := \{p : \underline{E} \leq E_p\}.$$

> 
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> (Dashed triangle delimits probability simplex.)

Gamble Space Representation

Problem Not all coherent sets of acceptable gambles can be (compactly) depicted by their intersection with a plane, as was done above.

Considerations

- Representation on a two-dimensional surface is possible by *Positive Homogeneity*.

Considerations (continued)

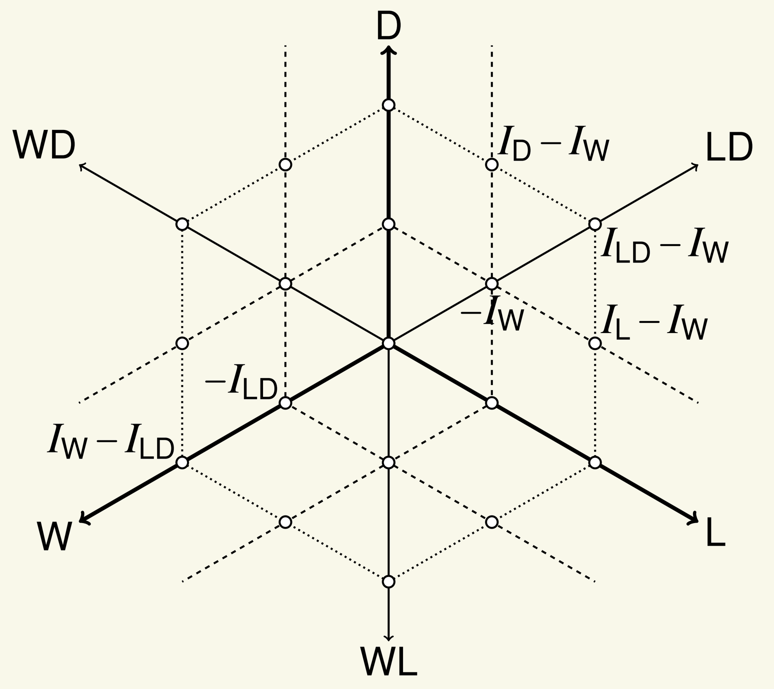
- The representation should be essentially invariant under permutations of Ω to avoid bias.
- The positive and negative octants do not need to be (faithfully) represented because of *Accepting Partial Gains* and *Avoiding Sure Loss*.
- To allow for intuitive exploration by the elicitee, the representation should provide a continuous deformation of the other octants.

These considerations lead us to a polar projection, where the poles are defined by the line of constant gambles. On the right, we show an example of a spherical such projection.

Reference value Anchor gamble payoffs by fixing their minimum value to -1, also to mitigate risk-aversion. Then the *stake* is equal to 1.

Surface to project The surface of the convex cone with apex $-I_{WDL} = (-1, -1, -1)$ and extreme rays $(1, 0, 0) \propto I_W$, $(0, 1, 0) \propto I_D$, and $(0, 0, 1) \propto I_L$.

Projection center $-I_{WDL} = (-1, -1, -1)$.

> 
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>
> - The dashed lines form the locus of the *contingent gambles*, i.e., those that are zero on the complement of the contingent event.
> - The dotted line indicates the locus of 'even' gambles, with the stake as maximum payoff.

> The octants visible in our representation; each is identified by putting a line over the negative components of its gambles:
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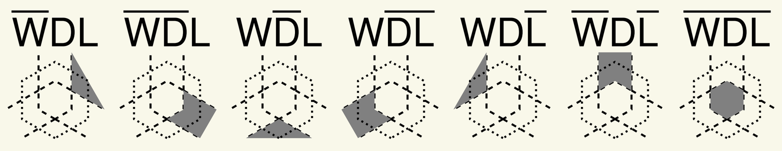
> 
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> ($\overline{\text{WDL}}$ represents a sure loss, WDL can be thought of as the line at infinity.)

> Rerepresentation of $\mathcal{D}$:
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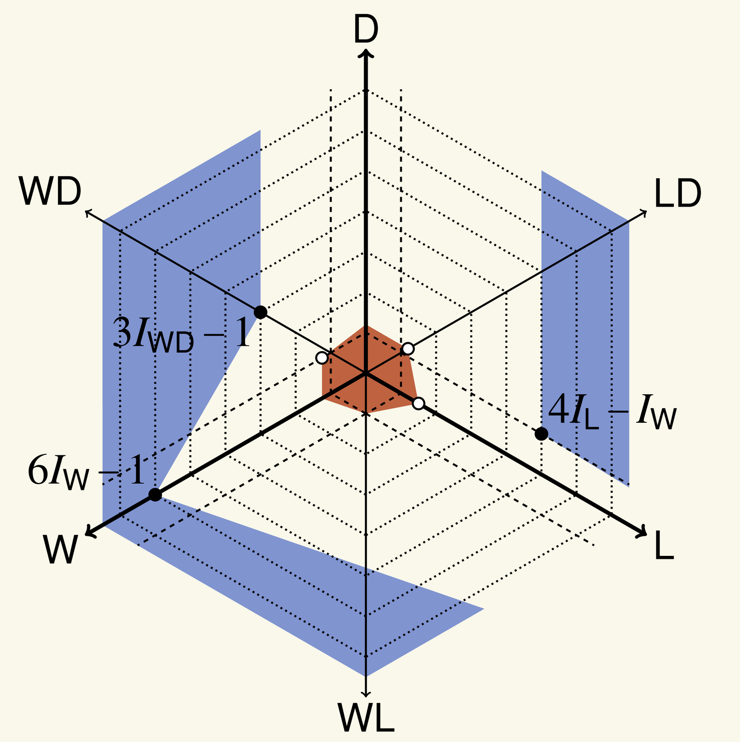
> 
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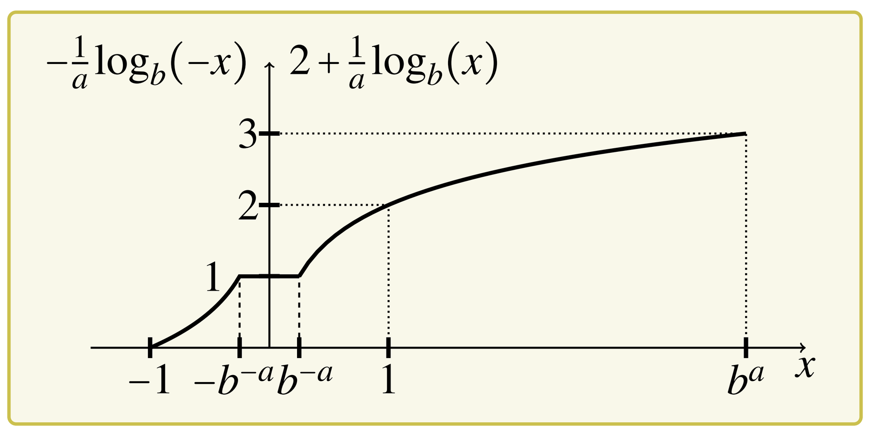
>
> - Dotted lines for the loci of gambles with maximum payoff one to six shown.
> - Added open convex polytope of '*rejected*' gambles that would cause a sure loss if one of them were to be assessed acceptable.

Range deficiency The linear scale used limits the range of possible payoff values.

Logarithmic scale Therefore, we use a custom scaling that is based on a 'saturating' logarithm.

> 
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> Logarithmically scaled rerepresentation of $\mathcal{D}$:
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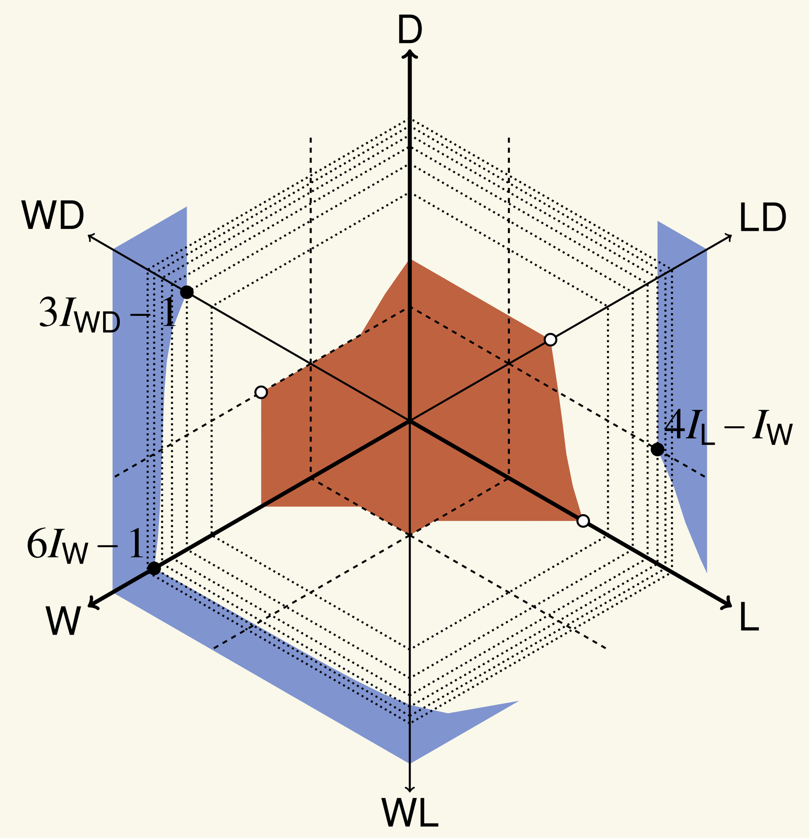
> 
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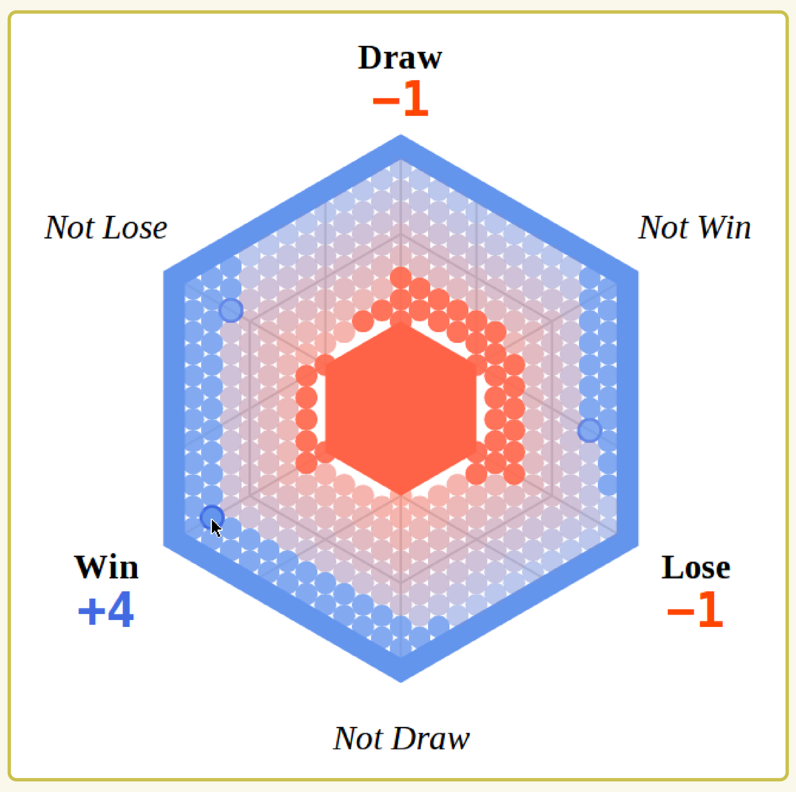
Implementation

Discretization

- Computing natural extension responsively.
- Show gamble values on hover, without a distracting number of significant digits.

> 
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> ($6I_W - 1$ replaced by $5I_W - 1 = (4, -1, -1)$.)

Challenge – responsive natural extension

Use 'inner' *propagation* routines:

- Pre-calculate the 'negation-dominance structure'; then acceptable and rejected gambles can be computed in pairs.
- Pre-calculate the 'dominance structure'; then we can recursively propagate gamble state.

… and an 'outer' *search* routine:

- The iteration over the accept (or reject) candidates is determined by a heuristic 'maximizing' propagation.

The Experiment

1982 World Cup (Walley's experiment)

- Eliciting lower and upper probabilities
- Pen & paper interface (?)
- 17 academic participants; 36 matches
- Assessments evaluated using the (6000!) possible pairwise 'fair' bets between them

2014 World Cup (Our experiment)

- Eliciting acceptable gambles
- On-line point-and-click interface ensuring coherence
- 80 mostly academic participants; 32 matches
- Assessments used in a betting pool; 100 'fair' gambles assigned in total

> A participant's played-match list at the end of the competition:
>
> 

Walley's fair bets Between a pair of participants:

- a pair of opposite 'simple' gambles,
- with equal nonnegative lower expectation.

Our fair bets Between all in a pool of participants:

- a set of gambles summing to the zero gamble,
- with equal nonnegative lower expectation,
- maximizing the sum of lower expectations (participants could be excluded from the bet).

(Involves a mixed-integer linear program.)

> An instance of the experiment's interface, including an assigned gamble:
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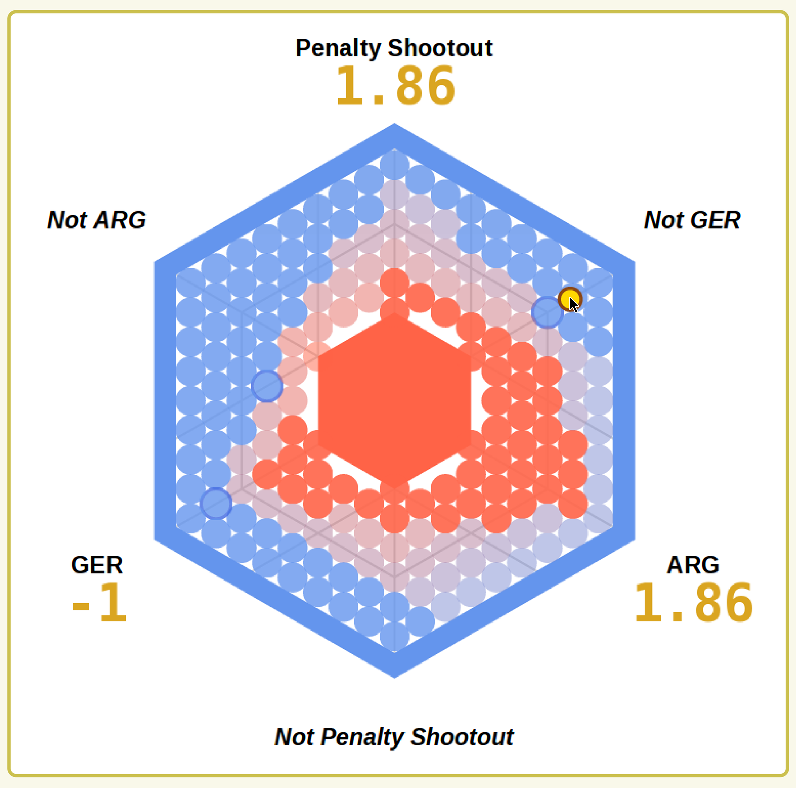
> 
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Results

Match assessments 194 in total.

Completeness Proportion of gambles being acceptable or rejected:

- A good 20% of assessments were complete. (The few participants who used complete models almost exclusively all had greater losses than winnings.)
- For the others, the degree of completeness varied over the whole range between just a few and all but a few marked gambles.

Selected dots per assessment

#gambles:	1	2	3	4	5	6	7	8
#assessments:	54	52	47	26	8	5	1	1

So participants usually kept things simple.

Selected gamble distribution Primarily gambles on the axes and contingent gambles were chosen, but not overwhelmingly so.

> Relative gamble selection frequency ($\propto$ dot area):
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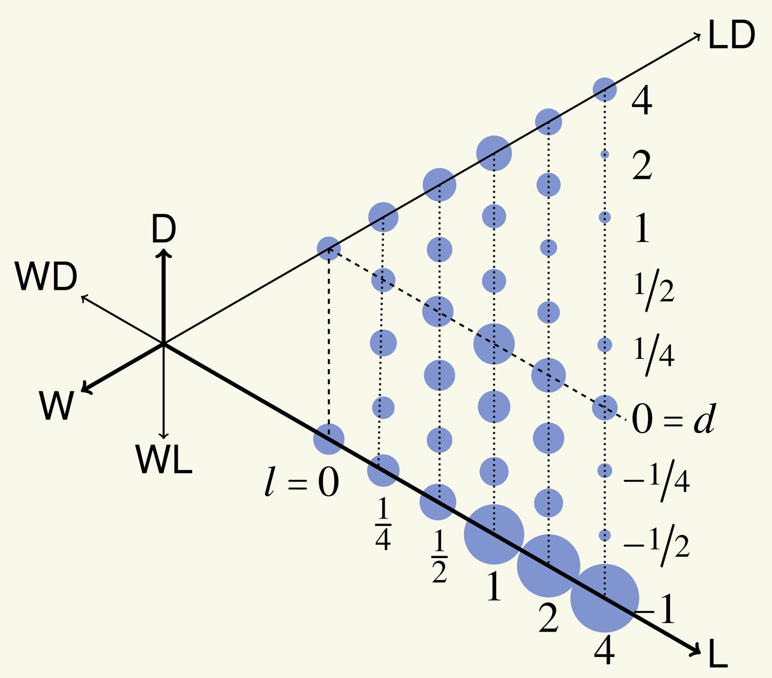
> 
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> Because of symmetry all gambles were mapped to the subregion $(-1, d, l)$. (Gamble $(-1, -1, 4)$, corresponds to 12.5%.)

Conclusions

- When given the option, people provide imprecise assessments.

> Credal sets for the final match, GER-ARG:
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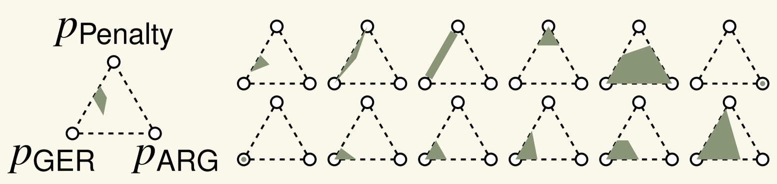
> 
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> The labeled simplex on the left contains the assessment shown earlier for this match.

- From participant feedback, we learned that the interface needs to be easier to understand.
- Often, many participants, mostly with relatively imprecise assessments, were excluded from bets. To improve feedback to users, the gamble assignment algorithm should be extended to be more inclusive.