Partial Partial Preference Order Orders

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Introduction

In this poster, I wish to present some preliminary ideas about the use of *exclusive disjunctions* in uncertainty modeling.

To start, I will use the following description of a collection of six-sided dice as a basis for a running example:

> *There are four faces, each present at least once: clubs ♣, spades ♠, diamonds ♢, and hearts ♡. A face only becomes visible after applying a drop of white wine to its side. There are at least three black faces. There are either more hearts than diamonds or an equal number of clubs and spades. A die is fair unless it has more black than white-faced sides, then each of the latter is equally more likely to land up than each of the former.*

Conceptual Approach

Possibility space Ω, restricted to observables.

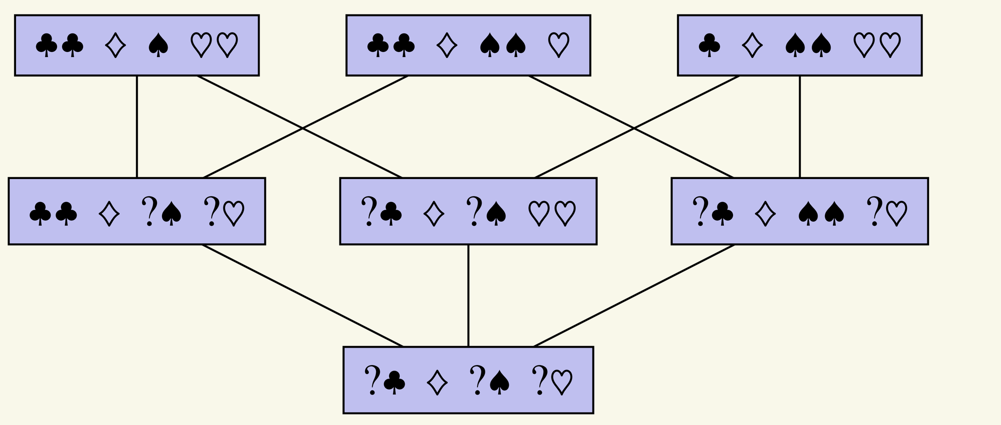
$$\Omega = \{\clubsuit, \diamondsuit, \spadesuit, \heartsuit\}$$

(The die variant should not be put into the possibility space.)

So a conscious decision is made about which exclusive disjunctions define the possibility space, and which not.

Partial order X, generated by the exclusive disjunctions.

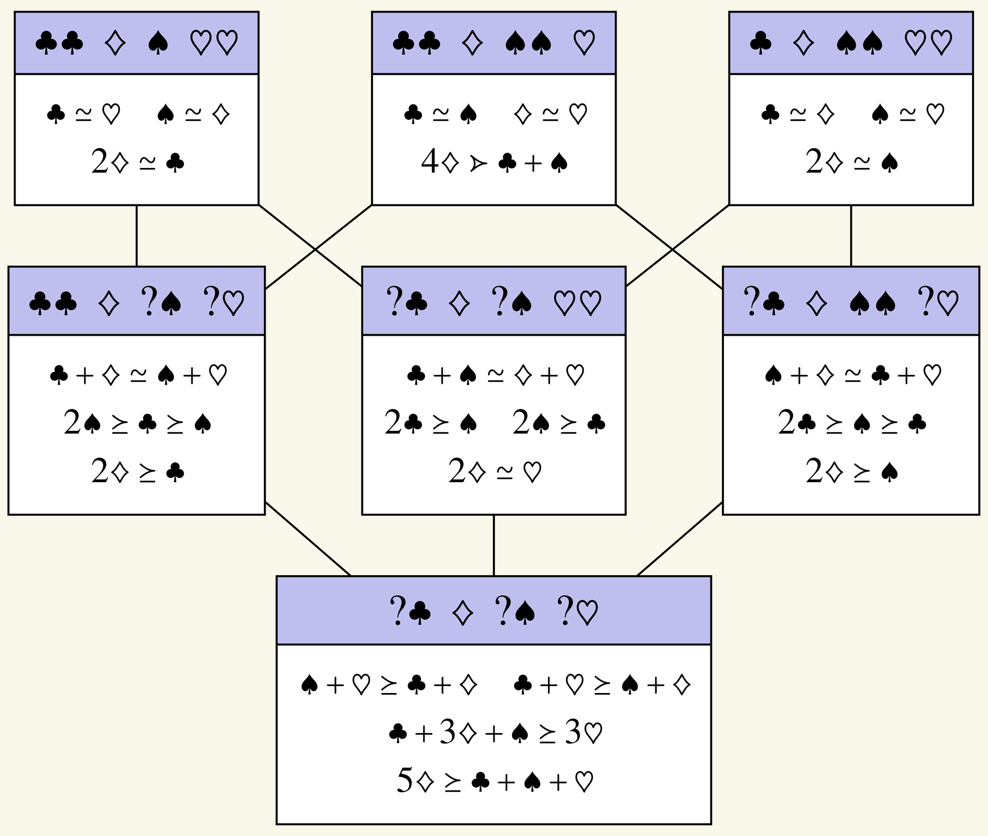
The Hasse diagram of the partial order for the dice example:



(The question mark '?' indicated that it is unknown whether there are one or two faces of the following type.)

Uncertainty models are attached to each element of the partial order X. (I am considering anything that is essentially a partial preference order of gambles, or equivalent to one—or a pair.) Each should 'reflect' the information common to its upset in X.

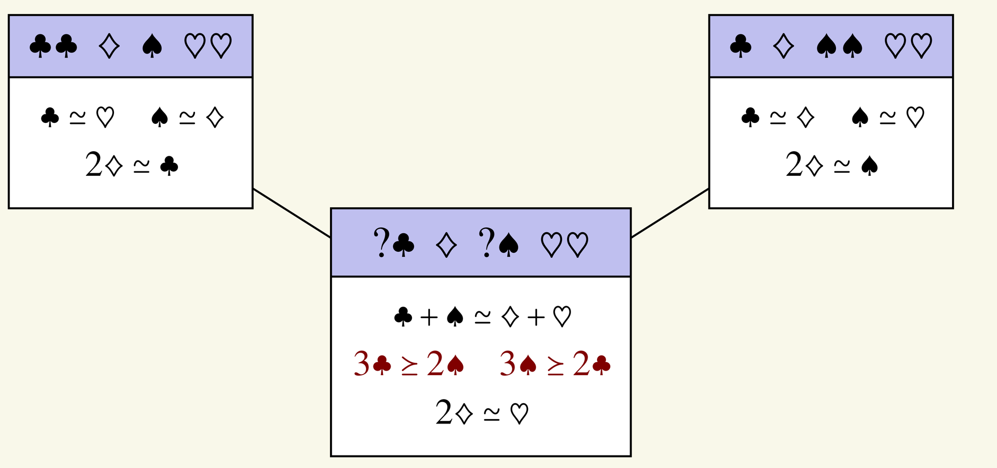
Here, we add partial preference relations:



- Each face stands for the corresponding indicator gamble.
- $\geq$ stands for (reflexive) acceptance.
- $\succ$ stands for (irreflexive) preference.
- $\simeq$ stands for indifference.

The uncertainty models may be more informative than the 'lower envelope' of their upset.

Zooming in on part of the above example, modifying one uncertainty model:



So X's non-maximal elements' uncertainty models can be seen as the result of some fusion or second order modeling operation.

Optimality criteria are attached to each element of the partial order X.

Maximality—non-domination in the partial preference order—is the most natural criterion.

$\underline{P}$-interval dominance can be used as a conservative approximation to maximality.

Maximin-of-maximal—those maximal gambles with maximum minimum payoff—is appropriate when guarantees on worst-case payoffs are desired.

$\underline{P}$-maximin—those gambles with maximum lower expected payoff—is appropriate when guarantees on expected worst-case payoffs are desired.

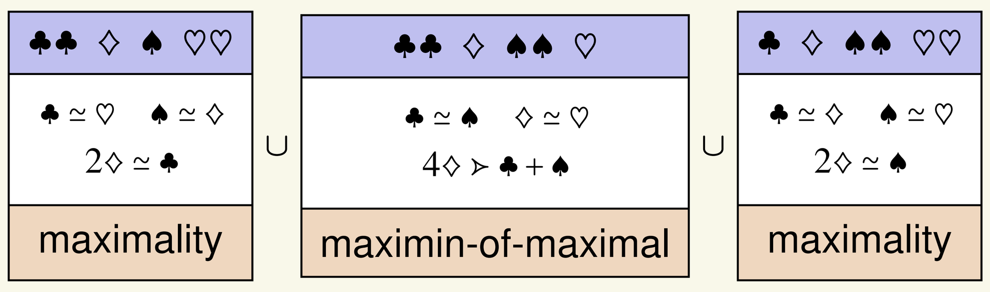
E-admissibility can be modeled as a partial order with, attached to its maximal elements, 'precise' uncertainty models and the maximality criterion.

Γ-maximin does not fit into our conceptual approach, when the set Γ of 'precise' uncertainty models is assumed to consist of those attached to the maximal elements of a partial order X. (Of course it is mathematically equivalent to attaching $\underline{P}$-maximin to the 'lower envelope' of Γ, which might not be present in the partial order, however.)

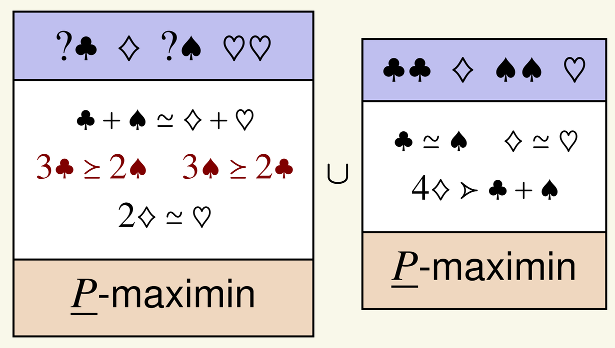
Choice functions C are derived as set functions of the *optimal elements* associated to each element of X, given some set of options. (A natural choice for this set function is taking the *union* over a *maximal antichain* of X.)

Option set:

$$O = \{\, 2\spadesuit + \diamondsuit - \heartsuit \,,\, 2\clubsuit + 2\diamondsuit - 2\heartsuit \,,\, 3\heartsuit - 2\spadesuit \,\}.$$



$$C(O) = \{3\heartsuit - 2\spadesuit\} \cup \{2\spadesuit + \diamondsuit - \heartsuit\} \cup \{2\spadesuit + \diamondsuit - \heartsuit\}.$$



$$C(O) = \{2\spadesuit + \diamondsuit - \heartsuit\} \cup \{2\spadesuit + \diamondsuit - \heartsuit, 2\clubsuit + 2\diamondsuit - 2\heartsuit\}.$$

Similar choices can be made based on different antichains by using appropriate optimality criteria.

We adapt an example from Seidenfeld et al.'s 2010 Synthese paper *Coherent choice functions under uncertainty* (§4): Two independent events (and their complements) are considered, A for the allergic state of a patient and S for the weather in New York City; so

$$\Omega = \left\{ \begin{array}{c|c} AS & A^cS \\ AS^c & A^cS^c \end{array} \right\}.$$

Two probability mass functions are used (each representing the opinion of a medical expert):

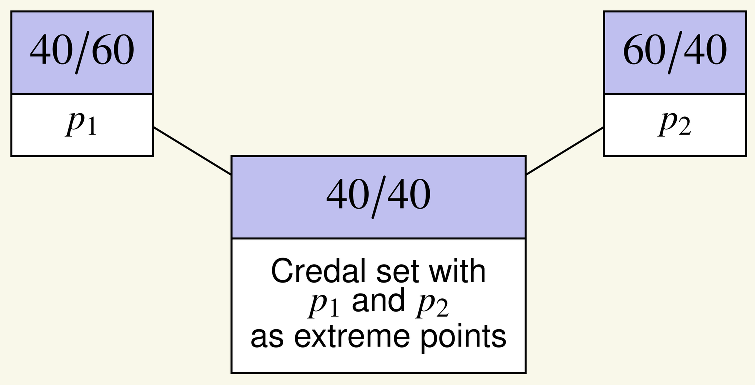
$$p_1 = \left(\begin{array}{c|c} .08 & .12 \\ .32 & .48 \end{array} \right), \qquad p_2 = \left(\begin{array}{c|c} .48 & .32 \\ .12 & .08 \end{array} \right).$$

Three options—treatment 'gambles'—are considered:

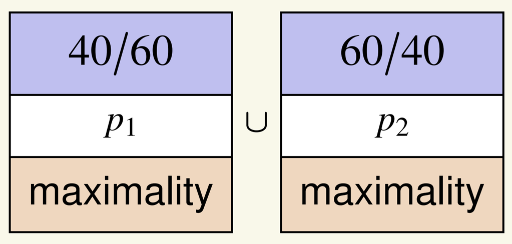
$$T_1 = \left(\begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array} \right), \qquad T_2 = \left(\begin{array}{c|c} 1 & 0 \\ 1 & 0 \end{array} \right), \qquad T_3 = \left(\begin{array}{c|c} .99 & -.01 \\ -.01 & .99 \end{array} \right),$$

where the last one is considered intuitively inadmissible.

Our partial order with uncertainty models attached:

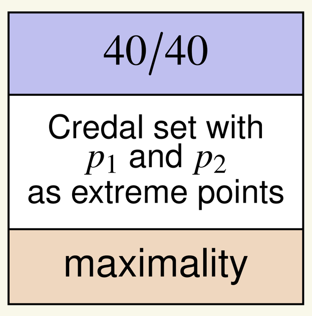


A solution:



$$C(\{T_1, T_2, T_3\}) = \{T_1, T_2\}.$$

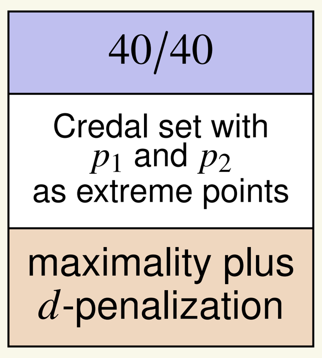
A non-solution:



$$C(\{T_1, T_2, T_3\}) = \{T_1, T_2, T_3\}.$$

Let d be a function that measures the distance of treatment gambles to the set of allergy-focused gambles (the span of T_1 and T_2). Gambles outside of this set can be considered to also be about betting on the weather, which is intuitively undesirable in this context.

An alternate solution:



$$C(\{T_1, T_2, T_3\}) = \{T_1, T_2\}.$$

Sets of probability measures

Operational assessments are seen as the basic building blocks of the uncertainty models: supremum acceptable buying prices, acceptable gambles, preferences between gambles,...

Probability measures are seen as a particular mathematical representation of some very special 'precise' uncertainty models.

Credal sets—*convex* sets of probability measures—are seen as alternative, (approximate) mathematical representations of the uncertainty models we consider.

Non-convex sets of probability measures are not regarded as uncertainty models here, but as special maximal antichains in some partial orders. *Even though they are often thought of as credal sets, I think 'sensitivity analysis' models belong here as well: being the 'true' probability is exclusive.*

Independence

Epistemic irrelevance is considered the basic notion.

Epistemic independence is its symmetrization.

Complete independence makes sense when all elements in a maximal antichain of X have 'precise' independent products attached.

Strong independence appears naturally in the 'lower envelopes' of partial orders with complete independence.

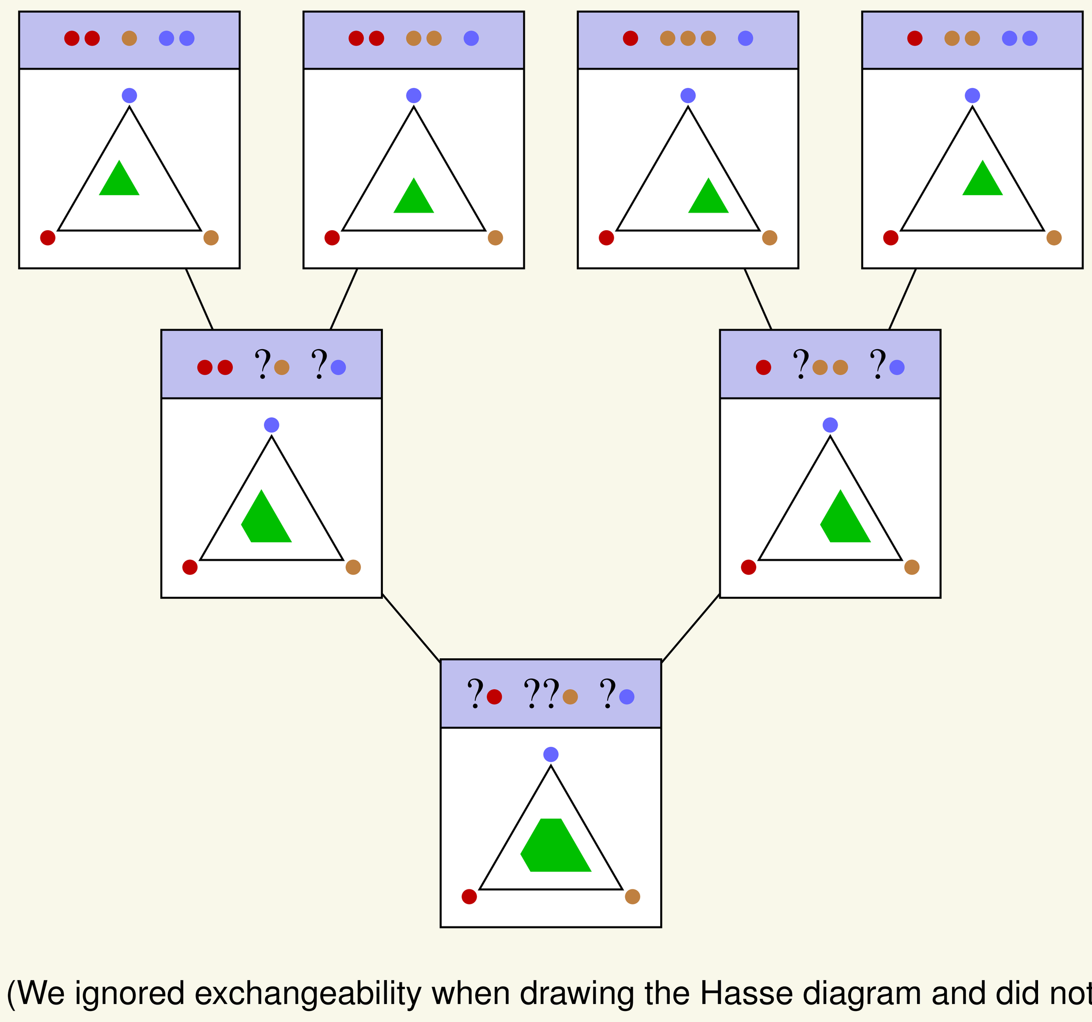
Outer approximations

Previously, we used uncertainty models in elements of X that were or dominated 'lower envelopes' of the uncertainty models in the upset. We can also consider situations where we use an outer approximation to such a lower envelope.

Consider a situation in which we are learning from a bag of marbles (•, •, and •) using an ID(M)M with $s = 2$. The *partially imprecise observation sequence* is

$$\bullet, \bullet, \bullet, \{\bullet, \bullet\}, \{\bullet, \bullet\}.$$

The resulting partial order, with uncertainty models attached:



(We ignored exchangeability when drawing the Hasse diagram and did not include elements due to the second ordering of imprecise observations.)

Logic

Algebraic normal form seems tailor-made for encoding antichains of uncertainty models.

The antichain in the first choice function example:

$$(\clubsuit \simeq \heartsuit)(\spadesuit \simeq \diamondsuit)(2\diamondsuit \simeq \clubsuit)$$
$$\oplus$$
$$(\clubsuit \simeq \spadesuit)(\diamondsuit \simeq \heartsuit)(4\diamondsuit \succ \clubsuit + \spadesuit)$$
$$\oplus$$
$$(\clubsuit \simeq \diamondsuit)(\spadesuit \simeq \heartsuit)(2\diamondsuit \simeq \spadesuit)$$

The antichain in the second choice function example:

$$(\clubsuit + \spadesuit \simeq \diamondsuit + \heartsuit)(3\clubsuit \geq 2\spadesuit)(3\spadesuit \geq 2\clubsuit)(2\diamondsuit \simeq \heartsuit)$$
$$\oplus$$
$$(\clubsuit \simeq \spadesuit)(\diamondsuit \simeq \heartsuit)(4\diamondsuit \succ \clubsuit + \spadesuit)$$

Questions

- How to combine such assessment expressions with expressions describing gamble subspaces for expressing marginal and contingent models? (Do we need to move to a modal logic?)
- Is the meaning of exclusive disjunction as we know it from logic too strong, considering our desire to use fused models in the partial orders?