Hyperbolic Systems with Random Set Coefficients

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Motivation

- Consider the problem

$$(1)\quad \begin{aligned} (\partial_t + \Lambda(x,t)\partial_x)u &= F(x,t)u + G(x,t), \quad (x,t)\in\mathbb{R}^2 \\ u(x,0) &= a(x), \quad x\in\mathbb{R} \end{aligned}$$

to be solved for unknown function $u=(u_1,\ldots,u_n)$.
  - $G=(G_1,\ldots,G_n), a=(a_1,\ldots,a_n)$
  - Λ and F are $(n\times n)$-matrix functions
  - **The coefficient matrix** Λ - real-valued and diagonal

$$\Lambda = diag(\lambda_1,\ldots,\lambda_n)$$

- Imprecise coefficients: each $\lambda_j, j=1,\ldots,n$ is given by
  - a random set
  - a random field (stochastic process in higher dimensions)
  - a random field with interval-valued parameters
- Applications: a prototype model for wave propagation in random media
- Methods: combining stochastics with
  - the method of characteristics
  - the algebras of Colombeau generalized functions

Method of characteristics

- The system (1) written componentwise for $j=1,\ldots,n$:

$$\begin{aligned} (\partial_t + \lambda_j(x,t)\partial_x)\,u_j &= F_j(x,t)u(x,t) + G_j(x,t), \quad (x,t)\in\mathbb{R}^2 \\ u_j(x,0) &= a_j(x), \quad x\in\mathbb{R} \end{aligned}$$

The integral curves obtained from the system

$$(2)\quad \begin{aligned} \tfrac{d}{d\tau}\gamma_j(x,t,\tau) &= \lambda_j(\gamma_j(x,t,\tau),\tau) \\ \gamma_j(x,t,t) &= x \end{aligned}$$

lead to the solution

$$u_j(x,t) = a_j(\gamma_j(x,t,0)) + \int_0^t F_j(\gamma_j(x,t,\tau),\tau,u(\gamma_j(x,t,\tau),\tau))d\tau$$

- It is also possible to establish the continuous dependence between coefficients $\lambda=(\lambda_1,\ldots,\lambda_n)$, characteristic curves $\gamma=(\gamma_1,\ldots,\gamma_n)$ and the solution u, neccessary for **the stochastic applications**:

$$(3)\quad \lambda\mapsto\gamma\mapsto u^\lambda$$

Random set coefficients

- $\lambda_j, j=1,\ldots,n$ are random sets on a probability space $(\Omega,\Sigma,\mathbb{P})$
  - Define

$$(4)\quad U(\omega)=\{u^l : l_j\in\lambda_j(\omega), j=1,\ldots,n\}, \omega\in\Omega.$$

  - Thanks to the **Fundamental Measurability Theorem**, it suffices to show that, for an arbitrary open set $O\subset\mathcal{C}(\mathbb{R}^2)$, the set

$$\{\omega : U(\omega)\cap O\neq\emptyset\}$$

is measurable.
  - After taking the sequence $\{l_j^k\}_{k\in\mathbb{N}}$, a **Castaing representation** of λ_j, we obtain

$$\{\omega : U(\omega)\cap O\neq\emptyset\} = \bigcup_{k\in\mathbb{N}}\left(u^{l_1^k,\ldots,l_n^k}\right)^{-1}(O)$$

  - This set is measurable, since (3) holds and we conclude that (4) defines a random set.

- $\lambda_j, j=1,\ldots,n$ are random fields with **Lipschitz continuous paths**
  - Since the map $\lambda\mapsto u^\lambda$ is *continuous*, the map $\omega\mapsto u^{\lambda(\omega)}$ is *measurable* as the composition of continuous and measurable functions.
  - The stochastic solution is a random field as well.
  - If additionally such random fields depend on **parameters** that are **interval-valued**, they can be considered to be random sets. Hence, the **solution is again a random set**.

- If the Lipschitz continuity is **not satisfied**, method of characteristics can no longer be used, because the solution of system (2) need not exist.

References:[1] M.Oberguggenberger, D.Rajter. Stochastic differential equations driven by generalized positive noise. *Publ. Inst.Math. Beograd, 77(91):7-19,2005*
[2] M. Oberguggenberger. *Multiplication of Distributions and Applications to Partial Differential Equations.* Pitman Res. Notes Maths. 259, Longman, Harlow, 1992.

The algebra of Colombeau generalized stochastic processes

- $(\Omega,\Sigma,\mathbb{P})$ - a probability space
- $\mathcal{E}(\mathbb{R}^n)$ - the space of nets $(X_\varepsilon)_\varepsilon, \varepsilon\in(0,1)$ of processes

$$X_\varepsilon : (0,1)\times\mathbb{R}^n\times\Omega\to\mathbb{R}^n$$

with almost surely smooth paths, such that:
  - $(t,\omega)\mapsto X_\varepsilon(t,\omega)$ is jointly measurable, for all $\varepsilon\in(0,1)$
  - $t\mapsto X_\varepsilon(t,\omega)$ belongs to $\mathcal{C}^\infty(\mathbb{R}^n)$, for all $\varepsilon\in(0,1)$ and almost all $\omega\in\Omega$.
- $\mathcal{E}_M^\Omega, \mathcal{N}^\Omega(\mathbb{R}^n)$ - spaces of nets of processes $(X_\varepsilon)_\varepsilon\in\mathcal{E}(\mathbb{R}^n),\ \varepsilon\in(0,1)$ with the following properties, respectively:
  - for almost all $\omega\in\Omega$, all compact $K\subset\mathbb{R}^n, \alpha\in\mathbb{N}_0^n$, there exist constants $N, C>0$ and $\varepsilon_0\in(0,1)$ such that

$$\sup_{t\in K}|\partial^\alpha X_\varepsilon(t,\omega)|\leq C\varepsilon^{-N}, \varepsilon\leq\varepsilon_0$$

  - for almost all $\omega\in\Omega$, all compact $K\subset\mathbb{R}^n, \alpha\in\mathbb{N}_0^n, b\in\mathbb{R}$, there exist $C>0$ and $\varepsilon_0\in(0,1)$ such that

$$\sup_{t\in K}|\partial^\alpha X_\varepsilon(t,\omega)|\leq C\varepsilon^{b}, \varepsilon\leq\varepsilon_0$$

- The differential algebra of **Colombeau generalized stochastic processes** is the factor algebra

$$\mathcal{G}^\Omega(\mathbb{R}^n) = \mathcal{E}_M^\Omega(\mathbb{R}^n)/\mathcal{N}^\Omega(\mathbb{R}^n).$$

Random fields with irregular paths

- Idea: smooth out the paths without Lipschitz property and solve

$$(5)\quad \begin{aligned} (\partial_t + \Lambda(x,t,\omega)\partial_x)\,U &= F(x,t)U(x,t)+G(x,t) \\ U|\{t=0\} &= A \end{aligned}$$

in the algebra of Colombeau generalized stochastic processes

- Appropriate conditions: initial data $A\in\mathcal{G}(\mathbb{R})$ is given
  - $F,G\in\mathcal{G}(\mathbb{R}^2),\ \Lambda\in\mathcal{G}^\Omega(\mathbb{R}^2)$
- F is *locally of logarithmic growth*, i.e. it has a representative $(f_\varepsilon)_\varepsilon, \varepsilon\in(0,1)$ such that:
  - for every compact $K\subset\mathbb{R}^2$ there exist $N\in\mathbb{N}, C,\varepsilon>0$ with

$$\sup_{(x,t)\in K}|f_\varepsilon(x,t)|\leq N\log\left(\tfrac{C}{\varepsilon}\right), 0<\varepsilon<\varepsilon_0$$

- Λ has a representative $(\Lambda_\varepsilon)_\varepsilon\in\mathcal{E}_M^\Omega(\mathbb{R}^2)$ with the property:
  - for almost all $\omega\in\Omega$ there exist $C,\varepsilon>0$ such that

$$\sup_{(x,t)\in\mathbb{R}^2}|\Lambda_\varepsilon(x,t,\omega)|\leq C, 0<\varepsilon<\varepsilon_0$$

- Λ has a representative $(\tilde\Lambda_\varepsilon)_\varepsilon\in\mathcal{E}_M^\Omega(\mathbb{R}^2)$ with the property:
  - for almost all $\omega\in\Omega$, for every compact $K\subset\mathbb{R}^2$ there exist $N\in\mathbb{N}, C,\varepsilon_0>0$ such that

$$\sup_{(x,t)\in K}\left|\partial_x\tilde\Lambda_\varepsilon(x,t,\omega)\right|\leq N\log\left(\tfrac{C}{\varepsilon}\right), 0<\varepsilon<\varepsilon_0$$

- Result: There is a unique solution $U\in\mathcal{G}^\Omega(\mathbb{R}^2)$ of the problem (5).