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An Idea on Consonant Conflicts between Belief Functions

Milan Daniel

Institute of Computer Science, Academy of Sciences of the Czech Republic

milan.daniel@cs.cas.cz

Introduction

General belief functions usually bear some internal conflict, which comes mainly from disjoint focal elements. Analogously there is often some conflict between two (or more) belief functions (BFs). This theoretical contribution introduces a new approach to conflicts of BFs. Conflicts between BFs are here considered independently of any combination rule and of any distance measure.

The suggested approach is based on consonant approximations of BFs in general; two important special cases based on consonant inverse pignistic and consonant inverse plausibility transformations are discussed. Their idea is based on our previous study of conflicts of BFs.

Theorem 1 Any BF (a,b) on 2-element frame of discernment Ω_2 is Dempster's sum of its **unique** *non-conflicting part* $(a_0, b_0) \in S_1 \cup S_2$ and of its **unique** *conflicting part* $(s, s) \in S$, which does not prefer any element of Ω_{2} , i.e. $(a,b) = (a_0, b_0) \bigoplus (s,s)$.

It holds true that $s = \frac{b(1-a)}{1-2a+b-ab+a^2} = \frac{b(1-b)}{1-a+ab-b^2}$ and $(a,b) = (\frac{a-b}{1-b}, 0) \oplus (s,s)$ for $a \ge b$; and similarly that $s = \frac{a(1-b)}{1+a-2b-ab+b^2} = \frac{a(1-a)}{1-b+ab-a^2}$ and $(a,b) = (0,\frac{b-a}{1-a}) \oplus (s,s)$ for $a \le b$.

Ω_n : existence of the unique non-conflicting part: **Theorem 2** (i) For any BF Bel defined on Ω_n there exists unique

consonant BF Bel_0 such that, $h(Bel_0 \oplus Bel_S) = h(Bel)$ for any BF Bel_S such that $Bel_0 \oplus U_n = U_n$.





Probabilistic approximations of belief functions were used in several previous approaches, e.g. pignistic probability in W. Liu's two-dimensional degree of conflict and in pignistic conflict and normalized plausibility of singletons in plausibility conflict.

Representation of Belief Functions by 2ⁿ-2 tuples

We can represent a BF by enumerantion of its $2^{n}-2$ basic belief masses: $\Omega_2 = \{\omega_1, \omega_2\}: (a, b) = (m(\{\omega_1\}, m(\{\omega_2\}), where m(\emptyset) = 0, m(\Omega_2) = 1 - a - b, \dots \}$ $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}: (d_1, d_2, d_3, d_{12}, d_{13}, d_{23}) = (m(\{\omega_1\}, m(\{\omega_2\}, m(\{\omega_3\}), (\{\omega_3\}), (\{\omega_3\}$ $m(\{\omega_1, \omega_2\}, m(\{\omega_1, \omega_3\}, m(\{\omega_2, \omega_3\}), \text{ where } m(\emptyset) = 0, m(\Omega_3) = 1 - \Sigma d_i - \Sigma d_{ii}.$ Graphically, we can represent them by the triangle in the case of Ω_2 , by 3D simplex in the case of quasi-Bayesian BFs on Ω_3 and by 6D simplex in the case of general BFs on Ω_3 .



Conflicts of Belief Functions

 Bel_1, Bel_2 , conflicting bbms: if $m_1(X) > 0$, $m_2(Y) > 0$, for $X \cap Y = \emptyset$, $m_{\bigcirc} = m_1 \bigcirc m_2 \quad m_{\bigcirc}(\emptyset) > 0$ $m_{O}(\mathcal{O}) = \Sigma_{X \cap Y = \mathcal{O}} m_1(X)m_2(Y)$ Shafer (76): weight of conflict between BFs . . . $m_{\bigcirc}(\emptyset)$. . . $log \frac{1}{1-m_{\bigcirc}(\emptyset)}$ $\Omega_6: m_i(\{\omega_i\}) = 1/6, \ m(\{\omega_i\}) = 1/36, \ m(\emptyset) = 5/6,$

(two numerically identical, but indepented BFs m_1, m_2) but, both intuitively and racionally no conflict between them

Hypothesis of existence of the unique indecisive conflicting part **Bel**_S of a BF Bel, such that $Bel = Bel_0 \bigoplus Bel_S$

Conflict between Belief Functions based of their Non-Conflicting Parts

Non-conflicting Parts of Belief Functions

 $Bel' \oplus Bel'' = Bel'_0 \oplus Bel'_S \oplus Bel''_0 \oplus Bel''_S =$

 $= (Bel'_{\theta} \oplus Bel''_{\theta}) \oplus (Bel'_{S} \oplus Bel''_{S})$

NO conflict inside *Bel'*₀, *Bel''*₀; *Bel'*_S, *Bel''*_S **non-conflicting with any BF** internal conflicts of *Bel'*, *Bel''* in *Bel'*, *Bel''*, *Bel''*, *conflict beween Bel'*, *Bel''* is between Bel'₀, Bel''₀.

Definition 3 Let *Bel'*, *Bel''* be two belief functions on *n*-element frame of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. Let Bel'_{θ} and Bel''_{θ} be their nonconflicting parts and m'_{0} , m''_{0} the related basic belief assignments (bbas). We define **conflict between BFs** *Bel'* and *Bel''* as $ncp\text{-}Conf(Bel',Bel'') = m_{Bel'_0} \otimes_{Bel''_0} (\emptyset) = (m'_0 \otimes m''_0)(\emptyset).$

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω_3
m'(X) :	0.375	0.100	0.225	0.10			0.20
m''(X) :	0.250	0.175	0.175	0.20	0.05	0.05	0.10
m''''(X) :	0.100	0.200		0.40	0.00	0.00	0.30

 $Pl_P' = (0.45, 0.20, 0.35), Pl_P'' = (0.40, 0.35, 0.25), Pl_P''' =$ (0.40,**0.45**,0.15), $m'_{0} = (10/45, 0, 0, 0, 15/45, 0; 20/45), m''_{0} = (1/8, 0, 0, 2/8, 0, 0; 5/8),$ $m'''_{0} = (0, 5/45, 0, 25/45, 0, 0; 15/45)$. Thus, Conf(Bel', Bel'') = 0 and $Conf(Bel'', Bel'''') = 5/45 \cdot 5/40 = 1/72$ and $Conf(Bel',Bel'''') = 10/45 \cdot 5/45 + 5/45 \cdot 15/45 = 5/81.$

NEW: an Idea of a Conflict Based on a Consonant Approximation

Probabilistic approximatons:

addition of some kind of conflicting information - internal conflicts of input belief are increased - we do not know, what is their effect to conflict "between"

New Idea:

- to use of inverse of probabilistic transformations

specially, consonant inverse approximations

- consonant, thus internally non-conflicting approximations (**0**,*b*) -some analogy to belief discounting (but without any discounting factor) uniquelly defined + probabilistic approx. kept 0 = (0,0)(1-b,0) (1,0) =⊤



consonant inverse contour approximation: iC: $Pl_P(iC(Bel)) = Pl_P(Bel)$

consonant inverse pignistic approximation: iBet: *BetP(iBet(Bel))* = *BetP(Bel)*

Several advantages: no internal conflicts, entire conflict of these approx. is 'between' no addional information nor internal conflict

Almond (95) $m_{\bigcirc}(\emptyset)$ is hardly interpretable as a conflict between BFs W. Liu (06) $m_{\bigcirc}(\emptyset)$ cannot be always interpretable as a conflict between

New approach: Daniel IPMU'10 total combinational conflict $m_{\bigcirc}(\emptyset)$ -- internal conflicts of input BFs -- conflict between input BFs 3 approaches to conflicts; conflict × difference, distance combinational, plausibility, comparative conflicts + pignistic conf. at SUM'13 + conflict based on non-conflicting parts at BELIEF'14

Plausibility Conflict

Definition 1 The internal plausibility conflict *Pl-IntC* of a BF *Bel* is defined as Pl-Int $C(Bel) = 1 - max_{\omega \in \Omega} Pl(\{\omega\})$, where Pl is the plausibility corresponding to Bel.

Conflict between (based on preference/oppositon of element of frame): indecisive BFs: $Pl_P(\omega_i) = 1/n$... all ω_i same support (no preference) in general:

some elements $Pl_P(\omega_i) > 1/n \dots \omega_i$ is supported/preferred some elements $Pl_P(\omega_{ii}) < 1/n \dots \omega_i$ is opposed

idea: same elements suported/opposed + same elements with max *Pl_P* value: no conflict between

conflicting elements: supported/preferred by *Bel_i* and opposed by *Bel_i* + set of max *Pl_P* elements (if disjoint sets)

 $\Omega_{PlC}(Bel_1, Bel_2)$... set of conflicting elements

Definition 2 Plausibility conflict between BFs *Bel*₁ and Bel₂ is defined $Pl-C(Bel_1, Bel_2) = min(Pl-C_0(Bel_1, Bel_2), (m_1 \odot m_2)(\emptyset)), \text{ where }$

Inverse Contour and Inverse Pignistic Consonant Conflicts

Definition 4 Inverse contour conflict is defined by formula iC- $Conf(Bel_1, Bel_2) = \sum_{iC} m_1(X)_{iC} m_2(Y),$ $X \cap Y = \emptyset$ where $X, Y \subseteq \Omega_n$. **Inverse pignistic conflict** is analogously defined by $iBet-Conf(Bel_1, Bel_2) = \sum_{iBet} m_1(X)_{iBet} m_2(Y),$ $X \cap Y = \emptyset$ where $X, Y \subseteq \Omega_n$. {ω₁,ω₂} **Non-conflictness of qBBFs:** (0.5,0.5,0,0,0,0) (0,0,0,1,0,0) (0,0,0,0,0,1) $[\omega_3]/(0,0,1,0,0,0)$ The situation is more complicated for general BFs:

 $iC-Conf(m_1,m_2) = 2/3 > 0 = iBet-Conf(m_1,m_2),$ **Example of different non-conflictness** of *iC-Conf* and *iBet-Conf*: $iC-Conf(m_1, m_3) = 0 < 1/4 = iBet-Conf(m_1, m_3).$ $m_1 = (1,0,0,0,0,0), m_2 = (1/3,0,0,0,0,2/3), m_3 = (0,0,1/2,1/2,0,0),$

Comparison of Consonant Conflicts with Previous Approaches

Equivalence of inverse contour conflict *iC-Conf* to *ncp-Conf*

Theorem 4 Consonant inverse contour conflict *iC-Con*f is equivalent to conflict between belief functions based on their non-conflicting parts *ncp-Conf*, i.e., for any pair of BFs *Bel'*, *Bel''* on general Ω_n it holds that iC-Conf(Bel',Bel'') = ncp-Conf(Bel',Bel'').

Basic properties (of both *iC-Conf* and *iBet-Conf*): (i) Non-negativity and boundary conditions: $0 \le Conf(Bel_1, Bel_2) \le 1$, $Conf(Bel_1, Bel_2) = 0 \text{ iff } \{\omega_i \mid Pl_1(\{\omega_i\}) \ge Pl_1(\{\omega_j\})\} \cap \{\omega_i \mid Pl_2(\{\omega_i\}) \ge Pl_2(\{\omega_j\})\} \neq \emptyset$ $Conf(Bel_1, Bel_2) = 1$ iff ... $(\exists X, Y \subseteq \Omega)(Bel_1(X) = 1 = Bel_2(Y) \& X \cap Y = \emptyset).$ (ii) Symmetry: $Conf(Bel_1, Bel_2) = Conf(Bel_2, Bel_1)$. (iii) Conf(Bel,Bel) = 0. A BF is not conflicting with itself. (iv) Conf(Bel, VBF) = 0. Vacuous BF is non-conflicting with any other BF.

(no distance used; triangle inequality does not hold true.)

Theorem 3 (i) Let Bel_1 , Bel_2 be any quasi Bayesian BFs on general finite frame of discernment Ω_n given by bbas m₁ and m₂.

For both conflicts *iC-Conf* and *iBet-Conf* between Bel_1 and Bel_2 it holds that

$$Conf(Bel_1, Bel_2) \le \sum_{X \cap Y = \emptyset} m_1(X)m_2(Y).$$

(ii) Equality $Conf(Bel_1, Bel_2) = \sum_{X \cap Y = \emptyset} m_1(X)m_2(Y)$ holds iff both BFs Bel_1 and Bel_2 are consonant (even not qBBF).

Comparison with Plausibility Conflict *Pl-C*

Same non-conflictness for both on Ω_3 Same non-conflictness for qBBF on Ω_n Same non-conflictness for *iC-Conf* on Ω_{p}



Combinational and Comparative Conflict

 $TotC(m_1, m_2) = m(\emptyset)$... total comb. conflict, $IntC(m_i)$... internal conflicts $TotC(m_1, m_2) \sim IntC(m_1) + IntC(m_2) + C(m_1, m_2)$ $C(m_1, m_2)$... combinational conflict between m_1 and m_2

 $1/2 TotC(m_1, m_1) \leq IntC(m_1) \leq TotC(m_1)$ $TotC(m_1, m_2) - (IntC(m_1) + IntC(m_1)) \leq C(m_1, m_2) \leq TotC(m_1, m_2)$

Specification of (part) of belief mass(es) to less focal element(s). m_1 and m_2 are comparatively non-conflicting iff they have common specialization.

Comparative conflict between BFs Bel₁ and Bel₂ is the least difference of more specified bbms derived from the input bbms m_1 and m_2 .

Liu's Degree of Conflict

 $cf(m_i, m_j) = (m_{\bigodot}(\emptyset), difBetP_{m_i}^{m_j})$ $difBetP_{m_i}^{m_j} = max_{A \subset \Omega}(|BetP_{m_i}(A) - BetP_{m_i}(A)|)$

Lemma 1 $difBetP_{m_i}^{m_j} = \frac{1}{2} \sum_{\omega \in \Omega} |BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\})|$

Counter example against Theorem 2 from **Belief'14** on Ω_3 : $m_1(\{\omega_1, \omega_2\}) = 0.7, m_1(\{\omega_1, \omega_3\}) = 0.3, \text{ and } m_2(\{\omega_2, \omega_3\}) = 1.0;$ There is $Pl_1 = (1.0, 0.7, 0.3, ...), iC_1 = (0.3, 0, 0, 0.4, 0, 0), ...$ $Pl_2 = (0, 1.0, 1.0, ...), iC_2 = (0, 0, 0, 0, 0, 1.0), \text{ thus } iC - Conf(m_1, m_2) = 0.3 \cdot 1.0$ = 0.3;

 $\Sigma_{X \cap Y = \mathcal{O}} m_1(X) m_2(Y) = 0 < 0.15 = iBet-Conf(m_1, m_2) < 0.30 = iC-Conf(m_1, m_2).$

Summary and Conclusions

A new definition of conflict between belief functions on a general frame of disc. Comparison with/to Harmanec's conflict (see ECSQARU'15). + comparison with previous approaches.

Simplification of *Pl-C* while keeping its nature, conflict size produced in a way compatible with combinational conflict, thus improvement of both approaches

This approach to conflicts increases general understanding of conflicts and BFs in general; it enables better combination of conflicting BFs in real applications.

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For more detail see [5].



Comparison with W. Liu's degree of conflict cf

 $cf(Bel_1,Bel_2) = (0,0)$ implies iBet-Conf=0; for qBBFs also *iBet-Conf=0*; for more detail see [5].



Combinational approach: some kind of improvement, but non-compatible in general: does not hold $Conf \leq m_{\bigcirc}(\emptyset)$.

Comparative approach:

based on completely different idea (specialization of BFs).

References

[1] M. Daniel. Conflicts within and between Belief Functions. In: IPMU 2010. LNAI, vol. 6178, pp. 696--705. Springer, Heidelberg, 2010. [2] M. Daniel. Non-conflicting and Conflicting Parts of Belief Functions. In: ISIPTA'11; Proc. of the 7th ISIPTA, pp 149--158. Sipta, 2011 [3] M. Daniel. Conflict between Belief Functions: a New Measure Based on Their Non-Conflicting Parts. In: Belief'14, LNAI vol. 8764, pp.321-330, Springer, Heidelberg, 2014.

[4] M. Daniel. A Comparison of Plausibility Conflict and of Degree Conflict Based on Amount of Uncertainty of Belief Functions. In: ECSQARU 2015, LNAI vol. 9161, pp. 440-450, Springer, Heidelberg, 2015.

[5] M. Daniel. An Introduction to Consonant Conflicts between Belief Functions. Technical Report ICS AS CR, Prague (In preparation).