An Idea on Consonant Conflicts between Belief Functions

Milan Daniel

Institute of Computer Science, Academy of Sciences of the Czech Republic

milan.daniel@cs.cas.cz

Introduction

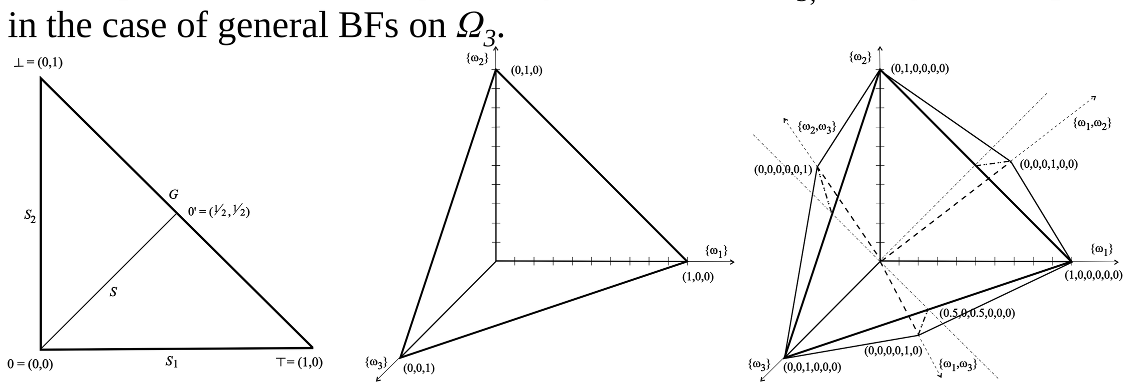
General belief functions usually bear some internal conflict, which comes mainly from disjoint focal elements. Analogously there is often some conflict between two (or more) belief functions (BFs). This theoretical contribution introduces a new approach to conflicts of BFs. Conflicts between BFs are here considered independently of any combination rule and of any distance measure.

The suggested approach is based on consonant approximations of BFs in general; two important special cases based on consonant inverse pignistic and consonant inverse plausibility transformations are discussed. Their idea is based on our previous study of conflicts of BFs.

Probabilistic approximations of belief functions were used in several previous approaches, e.g. pignistic probability in W. Liu's two-dimensional degree of conflict and in pignistic conflict and normalized plausibility of singletons in plausibility conflict.

Representation of Belief Functions by 2^n-2 tuples

We can represent a BF by enumeration of its 2^n-2 basic belief masses:
$\Omega_2 = \{\omega_1,\omega_2\}$: $(a,b) = (m(\{\omega_1\}, m(\{\omega_2\})$, where $m(\emptyset)=0$, $m(\Omega_2) = 1-a-b$,
$\Omega_3 = \{\omega_1,\omega_2,\omega_3\}$: $(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}) = (m(\{\omega_1\}, m(\{\omega_2\}, m(\{\omega_3\}), m(\{\omega_1,\omega_2\}, m(\{\omega_1,\omega_3\}, m(\{\omega_2,\omega_3\})$, where $m(\emptyset)=0$, $m(\Omega_3)=1-\Sigma d_i -\Sigma d_{ij}$.
Graphically, we can represent them by the triangle in the case of Ω_2, by 3D simplex in the case of quasi-Bayesian BFs on Ω_3, and by 6D simplex in the case of general BFs on Ω_3.



Conflicts of Belief Functions

Bel_1, Bel_2, conflicting bbms: if $m_1(X) > 0$, $m_2(Y) > 0$, for $X\cap Y = \emptyset$,
$m_{\circledcirc} = m_1 \circledcirc m_2$ $m_{\circledcirc}(\emptyset) > 0$ $\quad m_{\circledcirc}(\emptyset) = \Sigma_{X\cap Y=\emptyset}\, m_1(X)m_2(Y)$
Shafer (76): weight of conflict between BFs . . . $m_{\circledcirc}(\emptyset)$ . . . $log\frac{1}{1-m_{\circledcirc}(\emptyset)}$

Ω_6: $m_i(\{\omega_i\}) = 1/6$, $m(\{\omega_i\}) = 1/36$, $m(\emptyset) = 5/6$,
(two numerically identical, but indepented BFs m_1, m_2)
but, both intuitively and racionally no conflict between them

Almond (95) $m_{\circledcirc}(\emptyset)$ is hardly interpretable as a conflict between BFs
W. Liu (06) $m_{\circledcirc}(\emptyset)$ cannot be always interpretable as a conflict between

New approach: Daniel IPMU'10
total combinational conflict $m_{\circledcirc}(\emptyset)$ -- internal conflicts of input BFs
-- conflict between input BFs
3 approaches to conflicts; conflict × difference, distance
combinational, plausibility, comparative conflicts + pignistic conf. at SUM'13

+ **conflict based on non-conflicting parts** at BELIEF'14

Plausibility Conflict

Definition 1 The internal plausibility conflict *Pl-IntC* of a BF *Bel* is defined as $Pl\text{-}IntC(Bel) = 1 - max_{\omega\in\Omega}Pl(\{\omega\})$, where *Pl* is the plausibility corresponding to *Bel*.

Conflict between (based on preference/oppositon of element of frame):
indecisive BFs: $Pl\_P(\omega_i) = 1/n$ ... all ω_i same support (no preference)
in general:
some elements $Pl\_P(\omega_i) > 1/n$ ... ω_i is supported/preferred
some elements $Pl\_P(\omega_j) < 1/n$ ... ω_j is opposed

idea: same elements suported/opposed + same elements with max Pl_P value: **no conflict between**
conflicting elements: supported/preferred by Bel_i and opposed by Bel_j
+ set of max Pl_P elements (if disjoint sets)
$\Omega_{PlC}(Bel_1,Bel_2)$ ...set of conflicting elements

Definition 2 Plausibility conflict between BFs Bel_1 and Bel_2 is defined
$Pl\text{-}C(Bel_1,Bel_2) = min(\,Pl\text{-}C_0(Bel_1,Bel_2), (m_1 \circledcirc m_2)(\emptyset))$, where

$$Pl\text{-}C_0(Bel_1,Bel_2) = \sum_{\omega\in\Omega_{PlC}(Bel_1,Bel_2)} \frac{1}{2}|Pl\_P_1(\omega)-Pl\_P_2(\omega)|.$$

Combinational and Comparative Conflict

$TotC(m_1, m_2) = m(\emptyset)$... total comb. conflict, $IntC(m_i)$... internal conflicts
$TotC(m_1, m_2) \sim IntC(m_1) + IntC(m_2) + C(m_1, m_2)$
$C(m_1, m_2)$ ... combinational conflict between m_1 and m_2

$1/2\ TotC(m_1, m_1) \le IntC(m_1) \le TotC(m_1)$
$TotC(m_1, m_2) - (IntC(m_1) + IntC(m_1)) \le C(m_1, m_2) \le TotC(m_1, m_2)$

Specification of (part) of belief mass(es) to less focal element(s).
m_1 and m_2 are comparatively non-conflicting iff they have common specialization.
Comparative conflict between BFs Bel_1 and Bel_2 is the least difference of more specified bbms derived from the input bbms m_1 and m_2.

Liu's Degree of Conflict

$cf(m_i, m_j) = (m_{\circledcirc}(\emptyset), difBetP^{m_j}_{m_i})$
$difBetP^{m_j}_{m_i} = max_{A\subseteq\Omega}(|BetP_{m_i}(A) - BetP_{m_j}(A)|)$

Lemma 1 $difBetP^{m_j}_{m_i} = \frac{1}{2}\sum_{\omega\in\Omega}|BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\})|$

Non-conflicting Parts of Belief Functions

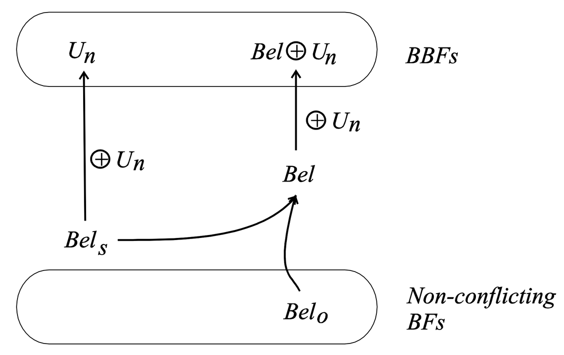
Theorem 1 Any BF *(a,b)* on 2-element frame of discernment Ω_2 is Dempster's sum of its **unique** *non-conflicting part* $(a_0,b_0) \in S_1\cup S_2$ and of its **unique** *conflicting part* $(s,s)\in S$, which does not prefer any element of Ω_2, i.e. $(a,b) = (a_0,b_0) \oplus (s,s)$.

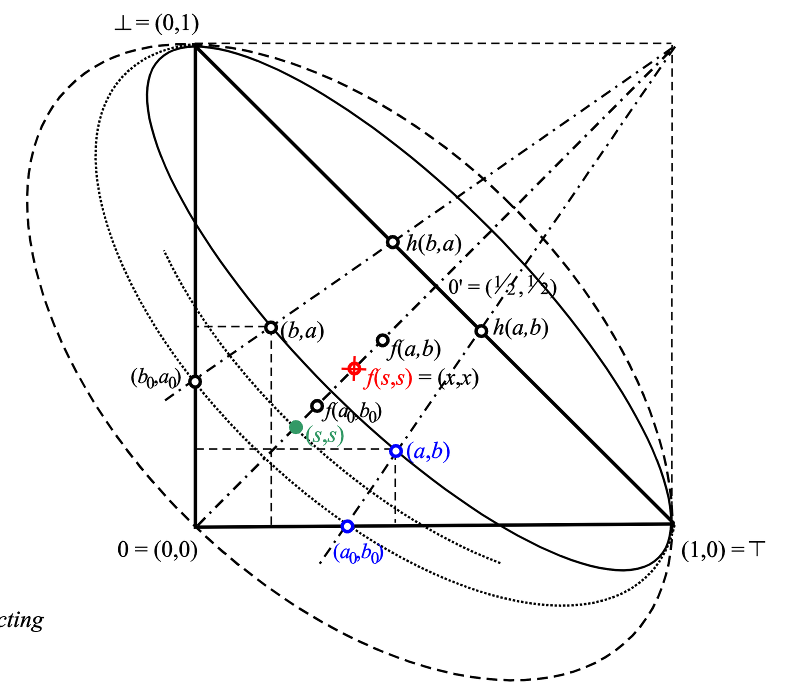
It holds true that $s = \frac{b(1-a)}{1-2a+b-ab+a^2} = \frac{b(1-b)}{1-a+ab-b^2}$ and $(a,b) = (\frac{a-b}{1-b},0) \oplus (s,s)$ for $a \ge b$; and similarly that $s = \frac{a(1-b)}{1+a-2b-ab+b^2} = \frac{a(1-a)}{1-b+ab-a^2}$ and $(a,b) = (0,\frac{b-a}{1-a}) \oplus (s,s)$ for $a \le b$.

Ω_n : existence of the unique non-conflicting part:
Theorem 2 (i) For any BF Bel defined on Ω_n there exists unique consonant BF Bel_0 such that, $h(Bel_0 \oplus Bel_S) = h(Bel)$ for any BF Bel_S such that $Bel_0 \oplus U_n = U_n$.

+

Hypothesis of existence of the unique indecisive conflicting part Bel_S of a BF *Bel*, such that $Bel = Bel_0 \oplus Bel_S$





Conflict between Belief Functions based of their Non-Conflicting Parts

$Bel' \oplus Bel'' = Bel'_0 \oplus Bel'_S \oplus Bel''_0 \oplus Bel''_S =$
$= (Bel'_0 \oplus Bel''_0) \oplus (Bel'_S \oplus Bel''_S)$
NO conflict inside Bel'_0, Bel''_0; Bel'_S,Bel''_S **non-conflicting with any BF**
internal conflicts of *Bel'*, *Bel''* in Bel'_S, Bel''_S ; conflict beween *Bel'*, *Bel''* is between Bel'_0, Bel''_0.

Definition 3 Let *Bel'*, *Bel''* be two belief functions on *n*-element frame of discernment $\Omega_n = \{\omega_1,\omega_2,...,\omega_n\}$. Let Bel'_0 and Bel''_0 be their non-conflicting parts and m'_0, m''_0 the related basic belief assignments (bbas). We define **conflict between BFs *Bel'* and *Bel''*** as

$$ncp\text{-}Conf(Bel', Bel'') = m_{Bel'_0 \circledcirc Bel''_0}(\emptyset) = (m'_0 \circledcirc m''_0)(\emptyset).$$

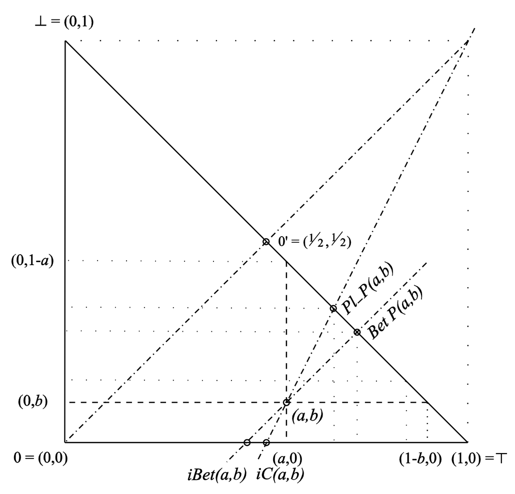
X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω_3
$m'(X)$:	0.375	0.100	0.225	0.10			0.20
$m''(X)$:	0.250	0.175	0.175	0.20	0.05	0.05	0.10
$m''''(X)$:	0.100	0.200		0.40	0.00	0.00	0.30

Pl_P' = (**0.45**,0.20,0.35), *Pl_P'' =* (**0.40**,0.35,0.25), *Pl_P'''' =* (0.40,**0.45**,0.15),
m'_0 = (**10/45**,0,0,0,15/45,0;20/45), m''_0 = (**1/8**,0,0,2/8,0,0;5/8),
m''''_0 = (0,**5/45**,0,25/45,0,0;15/45). Thus,
Conf(Bel',Bel'') = 0 and *Conf(Bel'',Bel'''') = 5/45 · 5/40 = 1/72* and
Conf(Bel',Bel'''') = 10/45 · 5/45 + 5/45 · 15/45 = 5/81.

NEW: an Idea of a Conflict Based on a Consonant Approximation

Probabilistic approximatons:
addition of some kind of conflicting information
- internal conflicts of input belief are increased
- we do not know, what is their effect to conflict "between"

New Idea:
- to use of inverse of probabilistic transformations specially, **consonant inverse approximations**
- consonant, thus internally non-conflicting approximations
-some analogy to belief discounting (but without any discounting factor)



consonant inverse contour approximation:
iC: *Pl_P(iC(Bel)) = Pl_P(Bel)*

consonant inverse pignistic approximation:
iBet: *BetP(iBet(Bel)) = BetP(Bel)*

Several advantages: no internal conflicts, entire conflict of these approx. is 'between' no addional information nor internal conflict uniquelly defined + probabilistic approx. kept

Inverse Contour and Inverse Pignistic Consonant Conflicts

Definition 4 Inverse contour conflict is defined by formula

$$iC\text{-}Conf(Bel_1, Bel_2) = \sum_{X\cap Y=\emptyset} {}_{iC}m_1(X)_{iC}m_2(Y),$$

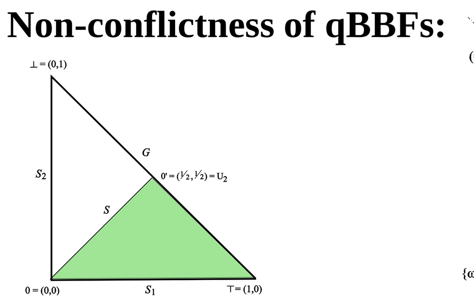
where $X,Y\subseteq\Omega_n$.

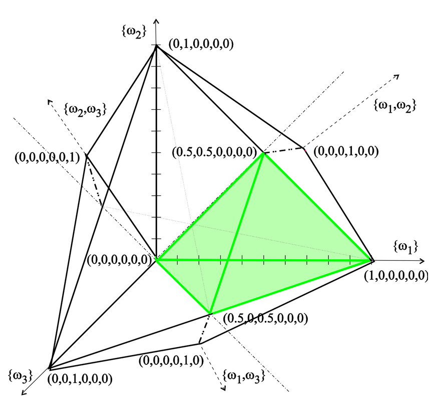
Inverse pignistic conflict is analogously defined by

$$iBet\text{-}Conf(Bel_1, Bel_2) = \sum_{X\cap Y=\emptyset} {}_{iBet}m_1(X)_{iBet}m_2(Y),$$

where $X,Y\subseteq\Omega_n$.

Non-conflictness of qBBFs:





Basic properties (of both *iC-Conf* and *iBet-Conf*):
(i) Non-negativity and boundary conditions: $0 \le Conf(Bel_1,Bel_2) \le 1$,
$Conf(Bel_1,Bel_2) = 0$ iff $\{\omega_i \mid Pl_1(\{\omega_i\}) \ge Pl_1(\{\omega_j\})\}\cap\{\omega_i \mid Pl_2(\{\omega_i\}) \ge Pl_2(\{\omega_j\})\} \neq \emptyset$
$Conf(Bel_1,Bel_2) = 1$ iff ... $(\exists X,Y\subseteq\Omega)(Bel_1(X) = 1 = Bel_2(Y)\ \&\ X\cap Y = \emptyset)$.
(ii) Symmetry: $Conf(Bel_1,Bel_2) = Conf(Bel_2,Bel_1)$.
(iii) $Conf(Bel,Bel) = 0$. A BF is not conflicting with itself.
(iv) $Conf(Bel,VBF) = 0$. Vacuous BF is non-conflicting with any other BF.
(no distance used; triangle inequality does not hold true.)

Theorem 3 (i) Let Bel_1, Bel_2 be any quasi Bayesian BFs on general finite frame of discernment Ω_n given by bbas m_1 and m_2.
For both conflicts *iC-Conf* and *iBet-Conf* between Bel_1 and Bel_2 it holds that

$$Conf(Bel_1, Bel_2) \le \sum_{X\cap Y=\emptyset} m_1(X)m_2(Y).$$

(ii) Equality $Conf(Bel_1,Bel_2) = \Sigma_{X\cap Y=\emptyset}\, m_1(X)m_2(Y)$ holds iff both BFs Bel_1 and Bel_2 are consonant (even not qBBF).

The situation is more complicated for general BFs:
Example of different non-conflictness of *iC-Conf* and *iBet-Conf*:
$m_1 = (1,0,0,0,0,0)$, $m_2 = (1/3,0,0,0,0,2/3)$, $m_3 = (0,0,1/2,1/2,0,0)$,
$iC\text{-}Conf(m_1,m_2) = 2/3 > 0 = iBet\text{-}Conf(m_1,m_2)$,
$iC\text{-}Conf(m_1,m_3) = 0 < 1/4 = iBet\text{-}Conf(m_1,m_3)$.

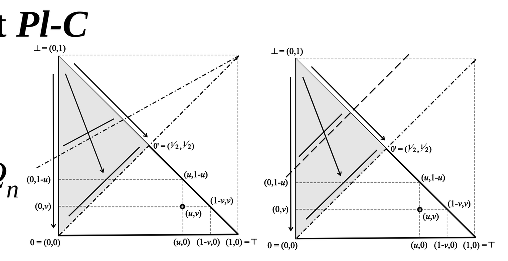
Comparison of Consonant Conflicts with Previous Approaches

Equivalence of inverse contour conflict *iC-Conf* to *ncp-Conf*

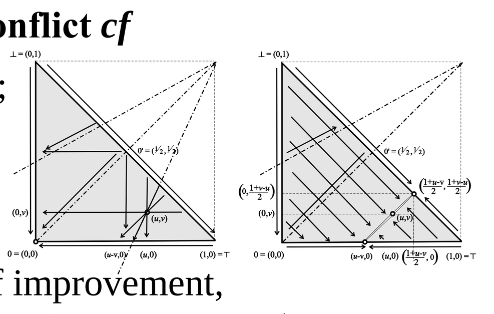
Theorem 4 Consonant inverse contour conflict *iC-Conf* is equivalent to conflict between belief functions based on their non-conflicting parts *ncp-Conf*, i.e., for any pair of BFs *Bel'*, *Bel''* on general Ω_n it holds that *iC-Conf(Bel',Bel'') = ncp-Conf(Bel',Bel'')*.

Counter example against Theorem 2 from **Belief'14** on Ω_3 :
$m_1(\{\omega_1, \omega_2\}) = 0.7$, $m_1(\{\omega_1, \omega_3\}) = 0.3$, and $m_2(\{\omega_2, \omega_3\}) = 1.0$;
There is $Pl_1 = (1.0,0.7,0.3,...)$, $iC_1 = (0.3,0,0,0.4,0,0)$, …
$Pl_2=(0,1.0,1.0,...)$, $iC_2 = (0,0,0,0,0,1.0)$, thus $iC\text{-}Conf(m_1,m_2) = 0 .3 \cdot 1.0 = 0.3$;
$\Sigma_{X\cap Y=\emptyset}m_1(X)m_2(Y)=0 < 0.15=iBet\text{-}Conf(m_1,m_2) < 0.30=iC\text{-}Conf(m_1,m_2)$.

Comparison with Plausibility Conflict *Pl-C*
Same non-conflictness for both on Ω_3
Same non-conflictness for qBBF on Ω_n
Same non-conflictness for *iC-Conf* on Ω_n
For more detail see [5].



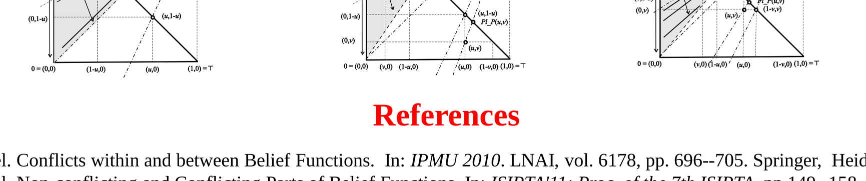
Comparison with W. Liu's degree of conflict *cf*
$cf(Bel_1,Bel_2)=(0,0)$ implies *iBet-Conf=0*;
for qBBFs also *iBet-Conf=0*;
for more detail see [5].



Combinational approach: some kind of improvement,
but non-compatible in general: does not hold $Conf \le m_{\circledcirc}(\emptyset)$.

Comparative approach:
based on completely different idea (specialization of BFs).

Comparison with/to Harmanec's conflict (see ECSQARU'15).



Summary and Conclusions

A new definition of conflict between belief functions on a general frame of disc. + comparison with previous approaches.

Simplification of *Pl-C* while keeping its nature, conflict size produced in a way compatible with combinational conflict, thus improvement of both approaches

This approach to conflicts increases general understanding of conflicts and BFs in general; it enables better combination of conflicting BFs in real applications.

Acknowledgement

The partial institutional support RVO: 67985807 is also acknowledged.

References

[1] M. Daniel. Conflicts within and between Belief Functions. In: *IPMU 2010*. LNAI, vol. 6178, pp. 696--705. Springer, Heidelberg, 2010.
[2] M. Daniel. Non-conflicting and Conflicting Parts of Belief Functions. In: *ISIPTA'11; Proc. of the 7th ISIPTA*, pp 149--158. Sipta, 2011.
[3] M. Daniel. Conflict between Belief Functions: a New Measure Based on Their Non-Conflicting Parts. In: Belief'14, LNAI vol. 8764, pp.321-330, Springer, Heidelberg, 2014.
[4] M. Daniel. A Comparison of Plausibility Conflict and of Degree Conflict Based on Amount of Uncertainty of Belief Functions. In: ECSQARU 2015, LNAI vol. 9161, pp. 440-450, Springer, Heidelberg, 2015.
[5] M. Daniel. An Introduction to Consonant Conflicts between Belief Functions. Technical Report ICS AS CR, Prague (In preparation).