On the Complexity of Propositional and Relational Credal Networks

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Hi

- From Engineering School, Universidade de São Paulo, Brazil.
- Collaboration with Denis:

- Interests in
  - Credal networks (this paper!).
  - Concepts of independence.
  - Full conditional probabilities and related stuff (the other paper!).

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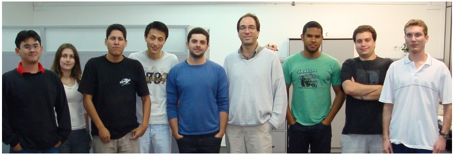

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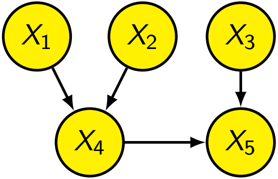
Goal

to study the relationship between

- specification language and
- complexity in Boolean credal networks.

Credal networks

- Directed acyclic graph, where each node is a random variable with associated "local" credal sets, with associated Markov condition.



- We focus on the *strong extension*:

$$\left\{ \mathbb{P} : \mathbb{P}(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i | \mathrm{pa}(X_i) = \pi_i) \right\}.$$

Complexity

- Marginal inference: $\overline{\mathbb{P}}(\mathbf{X}_Q = \mathbf{x}_Q | \mathbf{X}_E = \mathbf{x}_E) > \gamma$?

- $\mathsf{INF}_d(\mathcal{C})$: the inference problem for a class $\mathcal{C}$ of networks; $\mathsf{INF}_d^+(\mathcal{C})$ when evidence is positive (that is, $\{X = \text{true}\}$ is observed).

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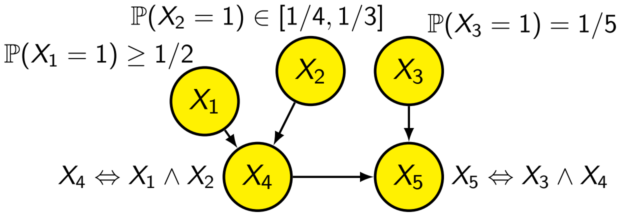
- $\mathrm{INF}_d(\mathcal{C})$: the inference problem for a class $\mathcal{C}$ of networks; $\mathrm{INF}_d^+(\mathcal{C})$ when evidence is positive (that is, $\{X = \mathrm{true}\}$ is observed).

  - In Bayesian networks: PP-complete problem.
  - In strong extensions: $\mathrm{NP}^{\mathrm{PP}}$-complete problem.

Specification framework: Propositional

- Associate, with each (Boolean) variable X, either
  - Equivalence $X \Leftrightarrow F(Y_1, \ldots, Y_m)$, where F is a sentence.
  - Assessment $\mathbb{P}(X = \text{true}) \in [\alpha, \beta]$.



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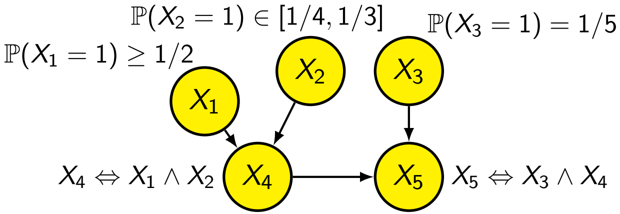

- Every propositional credal network can be specified this way.

Propositional credal networks: Results

- **Theorem**: $\mathsf{INF}_d^+(\mathsf{Prop}(\wedge, (\neg)))$ is polynomial.

- **Theorem**: $\mathsf{INF}_d^+(\mathsf{Prop}(\wedge, \vee, (\neg)))$ is $\mathsf{NP}^{\mathsf{PP}}$-complete.

Relational credal networks

- Extend: parameterized variables, with logical variables over (finite) domains.
- Example:

$$\begin{aligned}
\mathbb{P}(X_1(x) = 1) &\geq 1/2, \\
\mathbb{P}(X_2(x) = 1) &\in [1/4, 1/3], \\
\mathbb{P}(X_3(x, y) = 1) &= 1/5, \\
X_4(x) &\Leftrightarrow X_1(x) \wedge X_2(x), \\
X_5(x) &\Leftrightarrow \forall y : X_3(x, y) \wedge X_4(y).
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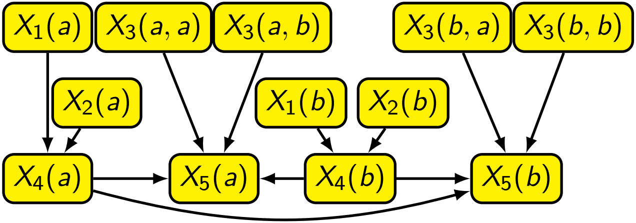
with domain $\mathcal{D} = \{a, b\}$:



Relational credal networks: Semantics

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  - *Coupled parameters*: for each $\gamma \in [1/2, 1]$,

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- Possible semantics:
  - *Coupled parameters*: for each $\gamma \in [1/2, 1]$,

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  - *Decoupled parameters*: for each $x \in \mathcal{D}$,

$$\mathbb{P}(X_1(x)) = \gamma \qquad \text{for each } \gamma \in [1/2, 1].$$

Relational credal networks: Results

- Explicit domain is given; inference is $\mathsf{INF}_d(\mathcal{C})$ with respect to grounded network where relations have bounded arity.
- Data complexity DINF_d: inference when model is fixed, and evidence and domain are inputs.

Relational credal networks: Results

- Explicit domain is given; inference is $\mathrm{INF}_d(\mathcal{C})$ with respect to grounded network where relations have bounded arity.
- Data complexity DINF_d: inference when model is fixed, and evidence and domain are inputs.

- **Theorem**: $\mathrm{INF}_d^+(\mathrm{FFFO})$ is $\mathrm{NP}^{\mathrm{PP}}$-complete both for coupled and decoupled parameters.
- **Theorem**: $\mathrm{DINF}_d(\mathrm{FFFO})$ is $\mathrm{NP}^{\mathrm{PP}}$-complete for decoupled parameters.
- **Theorem**: $\mathrm{DINF}_d(\mathrm{FFFO})$ is PP-complete for coupled parameters.

Conclusion

1. Specification language and complexity are inter-related.
2. Even for propositional networks, non trivial scenarios.
3. For relational networks, several open questions that go beyond Bayesian networks.