On the Number and Characterization of the Extreme Points of the Core of Necessity Measures on Finite Spaces

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Our Working Group

Background: Belief Functions (Shafer, 1976) and Necessity Measures (Dubois, Prade, 1988)

- Necessity measures on finite spaces treated as belief functions:
  - Basic probability assignment (bpa): $m : 2^\Omega \longrightarrow [0,1]$ with $\sum_{A \subseteq \Omega} m(A) = 1.$
  - Associated belief function $Bel : 2^\Omega \longrightarrow [0,1] : A \mapsto \sum_{B \subseteq A} m(B).$
  - **Focal sets** are all sets $A \subseteq \Omega$ with $m(A) > 0$.
  - Set of all focal sets is denoted with $\mathcal{F}(Bel)$.
- A necessity measure (on a finite space) is a map $N : 2^\Omega \longrightarrow [0,1]$ satisfying: $\forall A, B \in 2^\Omega : N(A \cap B) = \min\{N(A), N(B)\}$.
- It can be shown that (mathematically,) a necessity measure N is a special belief function, namely a belief function where all focal sets are nested (i.e.: $\forall A, B \in \mathcal{F}(N) : A \subseteq B$ or $B \subseteq A$).

The Core of a Belief Function

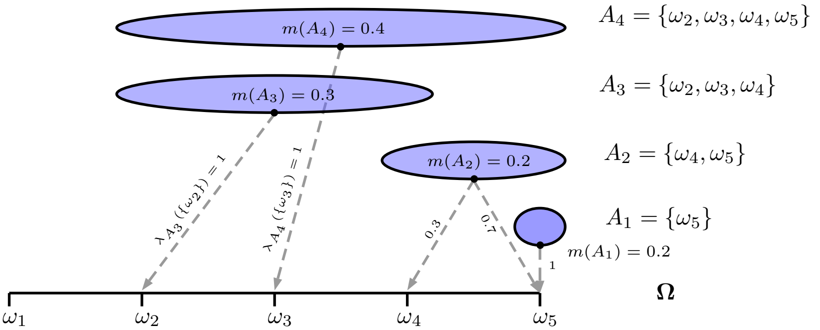
- Object of interest: the **core** of *Bel*:
  $\mathcal{M}(Bel) := \{P \in \mathscr{P}_n \mid \forall A \subseteq \Omega : P(A) \geq Bel(A)\}$.
- $\mathcal{M}(Bel)$ is a convex polytope that could be described by its extreme points $\text{ext}(\mathcal{M}(Bel))$.
- Aim: describing the extreme points of the core of a necessity measure (or more general, of a belief function).
- In general, describing the extreme points of the core of lower probabilities/previsions could be useful for:
  - decision making under partial prior knowledge
  - statistical hypothesis testing under imprecise probabilistic models
  - describing the core of convex games in the context of game theory
  - $\vdots$

The core $\mathcal{M}(Bel)$...

is obtained as all mass of focal sets A is transferred from the focal sets A to elements $\omega \in A$ (Chateauneuf, Jaffray, 1989):

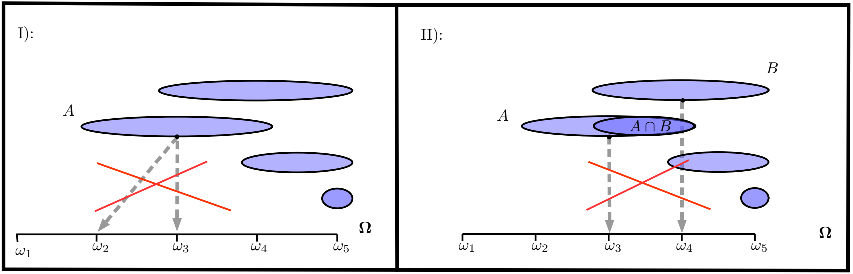
$$\forall p \in \mathcal{M}(Bel) \exists \lambda : \quad p(\{\omega\}) = \sum_{A \ni \omega} \lambda_A(\{\omega\}) \cdot m(A)$$

probability measure p — mass transfer λ — basic probability assignment m



The extreme points of the core satisfy:

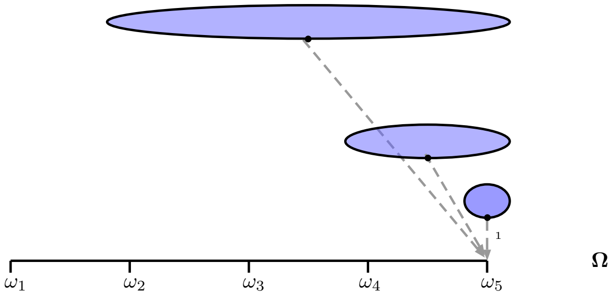
I) All mass $m(A)$ is transferred to exactly one state $\omega \in A$.
II) If all mass $m(A)$ is transferred to ω and all mass $m(B)$ is transferred to ω' and if $\{\omega, \omega'\} \subseteq A \cap B$ then $\omega = \omega'$.



For a necessity measure...

The extreme points can be described by looking at focal sets with increasing cardinality. **Observation:** The mass of a focal set A_{k+1} can be transferred

1. either somewhere outside the previous focal set A_k:
   For this one has $|A_{k+1} \backslash A_k|$ possibilities.
2. or somewhere into the previous focal set A_k:
   Then the mass has to be transferred to the same ω to which the mass of the previous focal set A_k is transferred.



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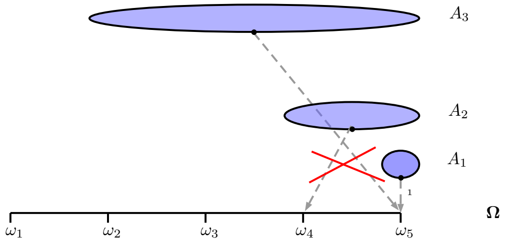

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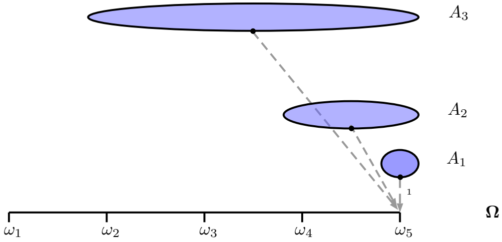

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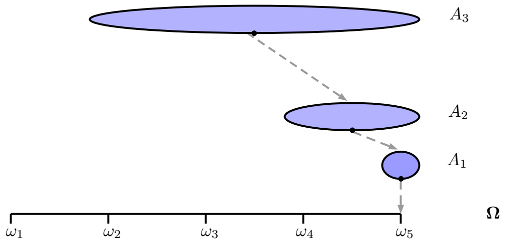

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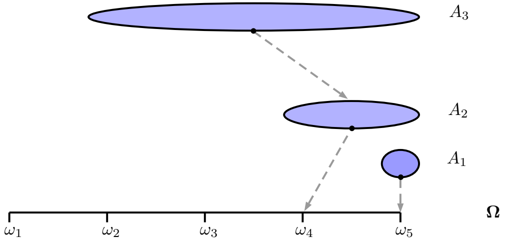

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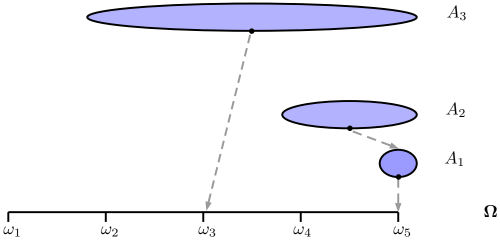

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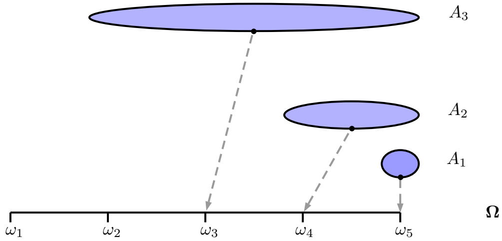

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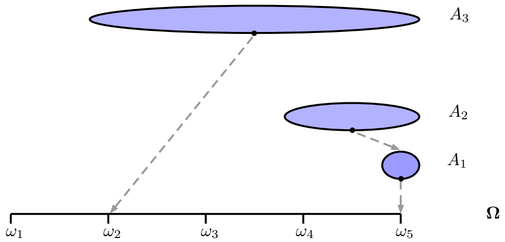

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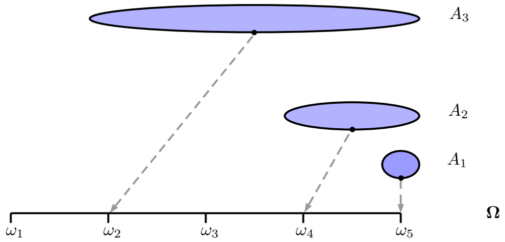


Main Theorem

Let N be a necessity measure with focal sets $\mathcal{F}(N) = \{A_1 \subset A_2 \subset \ldots \subset A_m\}$.

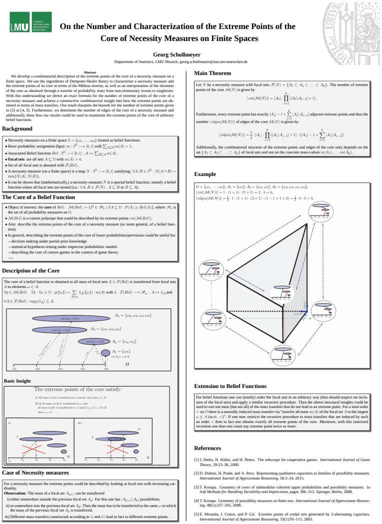
1. The number of extreme points of the core is given by

$$|\operatorname{ext}(\mathcal{M}(N))| = |A_1| \cdot \prod_{k=2}^{m} \left(|A_k \backslash A_{k-1}| + 1\right).$$

2. The number edges of the core $\mathcal{M}(N)$ is given by

$$|\operatorname{edges}(\mathcal{M}(N))| = \frac{1}{2} \cdot |A_1| \cdot \prod_{k=2}^{m} \left(|A_k \backslash A_{k-1}| + 1\right) \cdot \left(|A_1| - 1 + \sum_{k=2}^{m} |A_k \backslash A_{k-1}|\right).$$

3. Additionally, the combinatorial structure of the extreme points and edges of the core only depends on the set $\{A_1 \subset A_2 \subset \ldots \subset A_m\}$ of focal sets and not on the concrete mass-values $m(A_1), \ldots, m(A_m)$.



On the Number and Characterization of the Extreme Points of the Core of Necessity Measures on Finite Spaces

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Abstract

We develop a combinatorial description of the extreme points of the core of a necessity measure on a finite space. We use the ingredients of Dempster-Shafer theory to characterize a necessity measure and the extreme points of its core in terms of the Möbius inverse, as well as an interpretation of the elements of the core as obtained through a transfer of probability mass from non-elementary events to singletons. With this understanding we derive an exact formula for the number of extreme points of the core of a necessity measure and achieve a constructive combinatorial insight into how the extreme points are obtained in terms of mass transfers. Our result sharpens the bounds for the number of extreme points given in [5] or [4, 3]. Furthermore, we determine the number of edges of the core of a necessity measure and additionally show how our results could be used to enumerate the extreme points of the core of arbitrary belief functions.

Background

- Necessity measures on a finite space $\Omega = \{\omega_1, \ldots, \omega_n\}$ treated as belief functions:
- Basic probability assignment (bpa): $m : 2^\Omega \longrightarrow [0,1]$ with $\sum_{A \subseteq \Omega} m(A) = 1$.
- Associated Belief function $Bel : 2^\Omega \longrightarrow [0,1] : A \mapsto \sum_{B \subseteq A} m(A)$.
- **Focal sets** are all sets $A \subseteq \Omega$ with $m(A) > 0$.
- Set of all focal sets is denoted with $\mathcal{F}(Bel)$.
- A necessity measure (on a finite space) is a map $N : 2^\Omega \longrightarrow [0,1]$ satisfying: $\forall A, B \in 2^\Omega : N(A \cap B) = \min\{N(A), N(B)\}$.
- It can be shown that (mathematically,) a necessity measure N is a special belief function, namely a belief function where all focal sets are nested (i.e.: $\forall A, B \in \mathcal{F}(N) : A \subseteq B$ or $B \subseteq A$).

The Core of a Belief Function

- Object of interest: the **core** of Bel: $\mathcal{M}(Bel) := \{P \in \mathscr{P}_n \mid \forall A \subseteq \Omega : P(A) \geq Bel(A)\}$, where $\mathscr{P}_n$ is the set of all probability measures on Ω.
- $\mathcal{M}(Bel)$ is a convex polytope that could be described by its extreme points $\text{ext}(\mathcal{M}(Bel))$.
- Aim: describe the extreme points of the core of a necessity measure (or more general, of a belief function).
- In general, describing the extreme points of the core of lower probabilities/previsions could be useful for:
  - decision making under partial prior knowledge
  - statistical hypothesis testing under imprecise probabilistic models
  - describing the core of convex games in the context of game theory
  - ...

Description of the Core

The core of a belief function is obtained as all mass of focal sets $A \in \mathcal{F}(Bel)$ is transferred from focal sets A to elements $\omega \in A$:
$\forall p \in \mathcal{M}(Bel) \quad \exists \lambda : \forall \omega \in \Omega : p(\{\omega\}) = \sum_{A \ni \omega} \lambda_A(\{\omega\}) \cdot m(A)$ with $\lambda : \mathcal{F}(Bel) \longrightarrow \mathscr{P}_n : A \mapsto \lambda_A$ and $\forall A \in \mathcal{F}(Bel) : \text{supp}(\lambda_A) \subseteq A$.

Basic Insight

The extreme points of the core satisfy:

I) All mass $m(A)$ is transferred to exactly one state $\omega \in A$.

II) If all mass $m(A)$ is transferred to ω and all mass $m(B)$ is transferred to ω' and if $\{\omega, \omega'\} \subseteq A \cap B$ then $\omega = \omega'$.

Case of Necessity measures

For a necessity measure the extreme points could be described by looking at focal sets with increasing cardinality.
Observation: The mass of a focal set A_{k+1} can be transferred

i) either somewhere outside the previous focal set A_k: For this one has $|A_{k+1} \backslash A_k|$ possibilities.

ii) or somewhere into the previous focal set A_k: Then the mass has to be transferred to the same ω to which the mass of the previous focal set A_k is transferred.

iii) Different mass transfers constructed according to $i)$ and $ii)$ lead in fact to different extreme points.

Main Theorem

Let N be a necessity measure with focal sets $\mathscr{F}(N) = \{A_1 \subset A_2 \subset \ldots \subset A_k\}$. The number of extreme points of the core $\mathcal{M}(N)$ is given by

$$|\text{ext}(\mathcal{M}(N))| = |A_1| \cdot \prod_{i=2}^{k} (|A_i \backslash A_{i-1}| + 1).$$

Furthermore, every extreme point has exactly $|A_1| - 1 + \sum_{i=2}^{k} |A_i \backslash A_{i-1}|$ adjacent extreme points and thus the number $|\text{edges}(\mathcal{M}(N))|$ of edges of the core $\mathcal{M}(N)$ is given by

$$|\text{edges}(\mathcal{M}(N))| = \frac{1}{2} \cdot |A_1| \cdot \prod_{i=2}^{k} (|A_i \backslash A_{i-1}| + 1) \cdot (|A_1| - 1 + \sum_{i=2}^{k} |A_i \backslash A_{i-1}|).$$

Additionally, the combinatorial structure of the extreme points and edges of the core only depends on the set $\{A_1 \subset A_2 \subset \ldots \subset A_k\}$ of focal sets and not on the concrete mass-values $m(A_1), \ldots, m(A_k)$.

Example

$\Omega = \{\omega_1, \ldots, \omega_5\}$; $A_1 = \{\omega_5\}$; $A_2 = \{\omega_4, \omega_5\}$; $A_3 = \{\omega_2, \omega_3, \omega_4, \omega_5\}$.
$|\text{ext}(\mathcal{M}(N))| = 1 \cdot (1+1) \cdot (2+1) = 2 \cdot 3 = 6$,
$|\text{edges}(\mathcal{M}(N))| = \frac{1}{2} \cdot 1 \cdot (1+1) \cdot (2+1) \cdot (1 - 1 + 1 + 2) = \frac{1}{2} \cdot 6 \cdot 3 = 9$.

Extension to Belief Functions

For belief functions one can (totally) order the focal sets in an arbitrary way (that should respect set inclusion of the focal sets) and apply a similar recursive procedure. Then the above structural insights could be used to sort out most (but not all) of the mass transfers that do not lead to an extreme point. For a total order $<$ on Ω there is a naturally induced mass transfer via "transfer all mass $m(A)$ of the focal set A to the largest $\omega \in A$ (w.r.t. $<$)". If one now restricts the recursive procedure to mass transfers that are induced by such an order $<$ then in fact one obtains exactly all extreme points of the core. Moreover, with this restricted recursion one does not count any extreme point twice or more.

References

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