Imprecise random variables, random sets, and Monte Carlo simulation

Thomas Fetz, Michael Oberguggenberger

Unit for Engineering Mathematics
University of Innsbruck, Austria

Thomas.Fetz@uibk.ac.at
Michael.Oberguggenberger@uibk.ac.at

t.math

ISIPTA'15, Pescara, Italia

Research team

Unit of Engineering Mathematics
Faculty of Engineering Sciences
University of Innsbruck, Austria

Research team “imprecise probabilities”:

-  Michael Oberguggenberger (head of unit)
-  Thomas Fetz
-  Jelena Nedeljkovic
-  Martin Schwarz

Problem

Given

- Expensive input-output map $g : \mathbb{R}^n \to \mathbb{R} : x \to g(x)$.

  E.g. finite element computations (minutes or hours per computation).

- Family $\{X_\lambda\}_{\lambda \in \Lambda}$ of random variables modelling the uncertainty of variable x.

Aim

- Upper/lower probabilities that $g(x) \in B$.
- Upper/lower probabilities that $g(x) \leq y$ (upper/lower cumulative distribution functions).
- Upper/lower probabilities that $g(x) \leq 0$ (upper/lower probability of failure).

Two approaches

- Monte-Carlo simulation of $\{g(X_\lambda)\}_{\lambda \in \Lambda}$.
- Monte-Carlo simulation of the random set $\mathcal{X}$ generated by $\{g(X_\lambda)\}_{\lambda \in \Lambda}$.

Two approaches

1 Family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables

- Probability space $(\Omega,\Sigma,\mathrm{m})$.
- Family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables

$$X_\lambda : \Omega \to \mathbb{R} : \omega \to X_\lambda(\omega).$$

- Probability $P(X_\lambda \in B)$ for fixed X_λ:

$$P(X_\lambda \in B) = \int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\,\mathrm{dm}(\omega).$$

(for initial analysis we drop the map g)

2 Random set $\mathcal{X}$ based on $\{X_\lambda\}_{\lambda\in\Lambda}$

- Set-valued map $\mathcal{X} : \Omega \to \mathbb{R}$ defined by

$$\mathcal{X}(\omega) = \{X_\lambda(\omega) : \lambda \in \Lambda\}.$$

- $\mathcal{X}$ is a random set, if upper/lower inverses

$$\mathcal{X}^-(B) = \{\omega \in \Omega : \mathcal{X}(\omega) \cap B \neq \varnothing\},$$
$$\mathcal{X}_-(B) = \{\omega \in \Omega : \mathcal{X}(\omega) \subseteq B\}$$

are measurable subsets of Ω.

Two approaches

1 Family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables

- Probability space $(\Omega, \Sigma, \mathrm{m})$.
- Family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables

$$X_\lambda : \Omega \to \mathbb{R} : \omega \to X_\lambda(\omega).$$

- Probability $P(X_\lambda \in B)$ for fixed X_λ:

$$P(X_\lambda \in B) = \int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\, \mathrm{dm}(\omega).$$

(for initial analysis we drop the map g)

2 Random set $\mathcal{X}$ based on $\{X_\lambda\}_{\lambda\in\Lambda}$

- Set-valued map $\mathcal{X} : \Omega \to \mathbb{R}$ defined by

$$\mathcal{X}(\omega) = \{X_\lambda(\omega) : \lambda \in \Lambda\}.$$

- $\mathcal{X}$ is a random set, if upper/lower inverses

$$\mathcal{X}^-(B) = \{\omega \in \Omega : \mathcal{X}(\omega) \cap B \neq \varnothing\},$$
$$\mathcal{X}_-(B) = \{\omega \in \Omega : \mathcal{X}(\omega) \subseteq B\}$$

are measurable subsets of Ω.

Lower/upper probabilities for $\{X_\lambda\}_{\lambda\in\Lambda}$

$$\underline{P}(B) = \inf_{\lambda\in\Lambda} P(X_\lambda \in B) = \inf_{\lambda\in\Lambda} \int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\, \mathrm{dm}(\omega)$$

$$\overline{P}(B) = \sup_{\lambda\in\Lambda} P(X_\lambda \in B) = \sup_{\lambda\in\Lambda} \int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\, \mathrm{dm}(\omega)$$

Lower/upper probabilities for $\mathcal{X}$

$$\underset{\sim}{P}(B) = \mathrm{m}(\mathcal{X}_-(B)) = \int_\Omega \mathbb{1}_{\mathcal{X}(\omega)\subseteq B}\, \mathrm{dm}(\omega)$$

$$\widetilde{P}(B) = \mathrm{m}(\mathcal{X}^-(B)) = \int_\Omega \mathbb{1}_{\mathcal{X}(\omega)\cap B\neq\varnothing}\, \mathrm{dm}(\omega)$$

Two approaches

1 Family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables

- Probability space $(\Omega, \Sigma, \mathrm{m})$.
- Family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables

$$X_\lambda : \Omega \to \mathbb{R} : \omega \to X_\lambda(\omega).$$

- Probability $P(X_\lambda \in B)$ for fixed X_λ:

$$P(X_\lambda \in B) = \int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\, \mathrm{dm}(\omega).$$

(for initial analysis we drop the map g)

2 Random set $\mathcal{X}$ based on $\{X_\lambda\}_{\lambda\in\Lambda}$

- Set-valued map $\mathcal{X} : \Omega \to \mathbb{R}$ defined by

$$\mathcal{X}(\omega) = \{X_\lambda(\omega) : \lambda \in \Lambda\}.$$

- $\mathcal{X}$ is a random set, if upper/lower inverses

$$\mathcal{X}^-(B) = \{\omega \in \Omega : \mathcal{X}(\omega) \cap B \neq \varnothing\},$$
$$\mathcal{X}_-(B) = \{\omega \in \Omega : \mathcal{X}(\omega) \subseteq B\}$$

are measurable subsets of Ω.

Lower/upper probabilities for $\{X_\lambda\}_{\lambda\in\Lambda}$

$$\underline{P}(B) = \inf_{\lambda\in\Lambda} P(X_\lambda \in B) = \inf_{\lambda\in\Lambda} \int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\, \mathrm{dm}(\omega)$$

$$\overline{P}(B) = \sup_{\lambda\in\Lambda} P(X_\lambda \in B) = \sup_{\lambda\in\Lambda} \int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\, \mathrm{dm}(\omega)$$

Lower/upper probabilities for $\mathcal{X}$

$$\underset{\sim}{P}(B) = \mathrm{m}(\mathcal{X}_-(B)) = \int_\Omega \mathbb{1}_{\mathcal{X}(\omega)\subseteq B}\, \mathrm{dm}(\omega)$$

$$\widetilde{P}(B) = \mathrm{m}(\mathcal{X}^-(B)) = \int_\Omega \mathbb{1}_{\mathcal{X}(\omega)\cap B\neq\varnothing}\, \mathrm{dm}(\omega)$$

Theorem

$$\underset{\sim}{P} \le \underline{P} \le \overline{P} \le \widetilde{P}$$

$\mathcal{X}$ is more imprecise than $\{X_\lambda\}_{\lambda\in\Lambda}$!

- Probability space: $(\Omega,\Sigma,\mathrm{m}) = (\mathbb{R},\mathcal{B}(\mathbb{R}),\mathrm{m}),\quad \mathrm{m}(B) = \int_{\mathbb{R}} \mathbb{1}_{\omega\in B}\,\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\omega^2/2}\,\mathrm{d}\omega.$
- Family $\{X_{(\mu,\sigma)}\}_{(\mu,\sigma)\in\Lambda}$: $\quad X_{(\mu,\sigma)}(\omega) = \sigma\omega+\mu \implies X_{(\mu,\sigma)} \sim \mathcal{N}(\mu,\sigma^2).$
- $\Lambda = [\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}] = [-0.5,2]\times[1,2],\quad B = [1,2.5].$

$$\mathcal{X}(\omega) = \{X_\lambda(\omega) : \lambda\in\Lambda\} = [\underline{\mathcal{X}}(\omega),\overline{\mathcal{X}}(\omega)]$$

$$\underline{\mathcal{X}}(\omega) = \inf_{\substack{\mu\in[\underline{\mu},\overline{\mu}]\\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} X_{(\mu,\sigma)}(\omega) = \begin{cases} \overline{\sigma}\omega+\underline{\mu} & \omega<0 \\ \underline{\sigma}\omega+\underline{\mu} & \omega\geq 0 \end{cases}$$

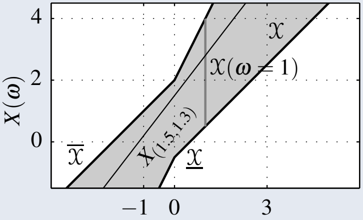
$$\overline{\mathcal{X}}(\omega) = \sup_{\substack{\mu\in[\underline{\mu},\overline{\mu}]\\ \sigma\in[\underline{\sigma},\overline{\sigma}]}} X_{(\mu,\sigma)}(\omega) = \begin{cases} \underline{\sigma}\omega+\overline{\mu} & \omega<0 \\ \overline{\sigma}\omega+\overline{\mu} & \omega\geq 0 \end{cases}$$

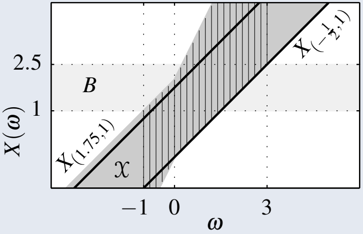
$$\underline{P}(B) = \inf_{(\mu,\sigma)\in\Lambda} P(X_{(\mu,\sigma)}\in B) = P(X_{(-0.5,1)}\in B) = 0.0655$$

$$\overline{P}(B) = \sup_{(\mu,\sigma)\in\Lambda} P(X_{(\mu,\sigma)}\in B) = P(X_{(1.75,1)}\in B) = 0.5467$$

$$\underset{\sim}{P}(B) = \mathrm{m}(\mathcal{X}_{-}(B)) = \mathrm{m}(\varnothing) = 0.0000$$

$$\widetilde{P}(B) = \mathrm{m}(\mathcal{X}^{-}(B)) = \mathrm{m}([-1,3]) = \Phi(3)-\Phi(-1) = 0.8400$$





Simulation of a family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables

1 Basic sample $x_1,\ldots,x_{N_{\text{samp}}}$

- Generate a sample $x_1,\ldots,x_{N_{\text{samp}}}$ which is distributed as a **basic random variable** X_*.
- Distribution of X_* should cover a greater range than a distribution of a single X_λ does.

2 N_{samp} function evaluations $g(x_k)$, $k=1,\ldots,N_{\text{samp}}$

- We compute $g(x_k)$ either using g directly or a cost saving surrogate model $\tilde{g}$.

3 Approximation of $P(g(X_\lambda)\le y)$

- Probability $P(g(X_\lambda)\le y)$ for fixed λ is computed by **reweighting** the original sample.
- Weights $w_k(\lambda)$ depending on parameters λ for reweighting the sample $x_1,\ldots,x_{N_{\text{samp}}}$ according to the distribution of X_λ:

$$w_k(\lambda)=\frac{f_{X_\lambda}(x_k)}{f_{X_*}(x_k)}\frac{1}{N_{\text{samp}}}=\frac{f_{\text{new}}(x_k)}{f_{\text{old}}(x_k)}\frac{1}{N_{\text{samp}}}$$

  where f_{X_λ} and f_{X_*} are strictly positive densities.
- $P(g(X_\lambda)\le y)$ for different X_λ **without additional function evaluations** of g:

$$P(g(X_\lambda)\le y)=\int_\Omega \mathbb{1}_{g(X_\lambda(\omega))\le y}\,\mathrm{dm}(\omega)\approx\sum_{k=1}^{N_{\text{samp}}}\mathbb{1}_{g(X_\lambda(\omega_k))\le y}\cdot w_k(\lambda)=\sum_{k=1}^{N_{\text{samp}}}\mathbb{1}_{g(x_k)\le y}\cdot w_k(\lambda).$$

4 Approximation of $\overline{P}(g \le y)$ and $\underline{P}(g \le y)$

For the computation of the upper/lower probabilities $\overline{P}(g \le y)$ and $\underline{P}(g \le y)$ we

- use a grid of representative parameter values λ_i,
- estimate the probabilities $P(g(X_{\lambda_i}) \le y)$ at the grid points λ_i by means of MC simulation
- and take the maximum/minimum value:

$$\overline{P}(g \le y) = \sup_{\lambda\in\Lambda} P(g(X_\lambda) \le y) \approx \max_{i=1,\dots,N_{\text{grid}}} P(g(X_{\lambda_i}) \le y) \approx \max_{i=1,\dots,N_{\text{grid}}} \sum_{k=1}^{N_{\text{samp}}} \mathbb{1}_{g(x_k)\le y} \cdot w_k(\lambda_i),$$

$$\underline{P}(g \le y) \approx \min_{i=1,\dots,N_{\text{grid}}} \sum_{k=1}^{N_{\text{samp}}} \mathbb{1}_{g(x_k)\le y} \cdot w_k(\lambda_i).$$

Effort: $N_{\text{grid}} \cdot N_{\text{samp}}$ reweightings, N_{samp} expensive function evaluations of g.

Simulation of a random set $\mathcal{X}$

1 Propagation of a random set through g

- $\mathcal{G}(\omega) = g(\mathcal{X}(\omega)) = \{g(X_\lambda(\omega))) : \lambda \in \Lambda\}$
- $\mathcal{G}(\omega) = [\underline{\mathcal{G}}(\omega), \overline{\mathcal{G}}(\omega)]$ random interval
- $\underline{\mathcal{G}}(\omega) = \min g(\mathcal{X}(\omega)),\ \overline{\mathcal{G}}(\omega) = \max g(\mathcal{X}(\omega))$

2 Cumulative distribution functions

- $\overline{F}(y) = \widetilde{P}(g \leq y),\ \underline{F}(y) = \underset{\sim}{P}(g \leq y)$
- $\overline{F}(y) = P\big((-\infty, y] \cap [\underline{\mathcal{G}}, \overline{\mathcal{G}}] \neq \varnothing\big) = P\big(\underline{\mathcal{G}} \leq y\big) = F_{\underline{\mathcal{G}}}(y)$
- $\underline{F}(y) = P\big([\underline{\mathcal{G}}, \overline{\mathcal{G}}] \subset (-\infty, y]\big) = P\big(\overline{\mathcal{G}} \leq y\big) = F_{\overline{\mathcal{G}}}(y)$

3 Algorithm for computing $\overline{F}(y)$

- Generate $\omega_1, \ldots, \omega_{N_{\text{samp}}}$ distributed as m.
- For each ω_n, estimate $\underline{\mathcal{G}}(\omega_n) \approx \min_i g(X_{\lambda_i}(\omega_n))$ using grid points $\lambda_1, \ldots, \lambda_{N_{\text{grid}}}$ on Λ.
- $\overline{F}(y) \approx \sum_{k=1}^{N_{\text{samp}}} \mathbb{1}_{\underline{\mathcal{G}}(\omega_k) \leq 0} \cdot \frac{1}{N_{\text{samp}}}$.

Effort: $N_{\text{grid}} \cdot N_{\text{samp}}$ expensive evaluations of g.

4 Cost saving methods, approximation of g by a **surrogate model** $\widetilde{g}$

Starting point: Collocation points $x_j, j = 1, \ldots, N_{\text{coll}}$, in $\mathbb{R}^n$ and N_{coll} evaluations $y_j = g(x_j)$.

Two levels are at hand: $\Omega \xrightarrow{X_\lambda} \mathbb{R}^n \xrightarrow{g} \mathbb{R}$.

(A) Surrogate model $\widetilde{g}$ of the map $g : \mathbb{R}^n \to \mathbb{R}$:

To obtain the lower bound $\underline{\mathcal{G}}$ we replace g by $\widetilde{g}$: $\underline{\mathcal{G}}(\omega_n) \approx \min_{i=1,\ldots,N_{\text{grid}}} \widetilde{g}(X_{\lambda_i}(\omega_n))$.

Effort: One surrogate model $\widetilde{g}$,

$N_{\text{grid}} \cdot N_{\text{samp}}$ cheap evaluations of $\widetilde{g}$ and N_{coll} expensive evaluations of g.

(B) Surrogate models $\widetilde{g}_i$ of maps $\Omega \to g \circ X_{\lambda_i}$:

- Collocation points x_j are pulled back to Ω.
- For each λ_i and x_j, we get a collocation point $\omega_{ij} = X_{\lambda_i}^{-1}(x_j)$ in Ω.
- Clearly, $y_j = g(X_{\lambda_i}(\omega_{ij})) = g(x_j)$ for every i. Then $\underline{\mathcal{G}}(\omega_n) \approx \min_{i=1,\ldots,N_{\text{grid}}} \widetilde{g}_i(\omega_n)$.

Effort: N_{grid} surrogate models $\widetilde{g}_i$,

N_{samp} cheap evaluations of $\widetilde{g}_i$ for each i and N_{coll} expensive evaluations of g.

Advantage of surrogate models $\widetilde{g}_i$ on Ω:

- Use of orthogonal polynomials with respect to the measure m.
- In the Gaussian case it means Hermite expansion.

Poster

Please visit our poster for more details and numerical examples!

Thank you for your attention!