Weak Consistency for Imprecise Conditional Previsions

Renato Pelessoni   Paolo Vicig

DEAMS 'B. de Finetti'
University of Trieste, Italy

ISIPTA'15, Pescara, 20-24 July 2015

Outline

- Main purpose: introduce general consistency concepts for lower conditional previsions, weaker than (Williams-)coherence and convexity, preserving a unitary approach.
- Starting point: generalise n-coherent unconditional previsions in Walley (1991)
  - Focus on (centered) 2-convex and 2-coherent conditional lower previsions, as these are the *most significant and general* models within n-convexity and n-coherence.
  - Study their characterisation and main properties, in particular those shared with the stronger notion of coherence.
  - → They satisfy the *GBR* and have a (2-coherent or 2-convex) *natural extension*.
  - Characterise 2-convexity and 2-coherence in terms of *desirability*.
  - 2-convex uncertainty models: conditional capacities, niveloids,...

Coherence and convexity

Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a conditional lower prevision.
$\forall n \in \mathbb{N}_0$, $X_0|B_0, \ldots, X_n|B_n \in \mathcal{D}$, $s_0 \in \mathbb{R}$, $s_1, \ldots, s_n \geq 0$ define

$$S(\underline{s}) = \bigvee \{B_i : s_i \neq 0, i = 0, \ldots, n\},$$
$$\underline{G} = \sum_{i=1}^{n} s_i B_i (X_i - \underline{P}(X_i|B_i)) - s_0 B_0 (X_0 - \underline{P}(X_0|B_0)),$$

$\forall \underline{G}$ s.t. $S(\underline{s}) \neq \emptyset$, let $\sup\{\underline{G}|S(\underline{s})\} \geq 0$.

- $s_0 \geq 0 \Rightarrow \underline{P}$ is *coherent* (Williams, 1975).
- $\sum_{i=1}^{n} s_i = 1 = s_0$ (convexity constraint) $\Rightarrow \underline{P}$ is *convex* (Pelessoni, Vicig, 2005)
- Convexity + $0|B \in \mathcal{D}$ and $\underline{P}(0|B) = 0$, $\forall X|B \in \mathcal{D}$ $\Rightarrow \underline{P}$ is *centered convex (C-convex)*

Weakening coherence and convexity

Basic idea: introduce constraints on n.
Most prominent case: $n = 1$ (i.e. 2 addends in $\underline{G}$).

$$\sup\{\underline{G}|S(\underline{s})\} \geq 0 \text{ (with } S(\underline{s}) \neq \emptyset)$$

- $\forall \underline{G}$ s.t. $n = 1$ $\Rightarrow$ $\underline{P}$ is *2-coherent*.
- $\forall \underline{G}$ s.t. $n = 1,\ s_0 = s_1 = 1$, $\Rightarrow$ $\underline{P}$ is *2-convex*.
- 2-convexity $+\ 0|B \in \mathcal{D}$ and $\underline{P}(0|B) = 0$, $\forall X|B \in \mathcal{D}$ $\Rightarrow$ $\underline{P}$ is *centered 2-convex*.

Features of 2-convex lower previsions

Let $\underline{P} : \mathcal{D} \rightarrow \mathbb{R}$ be a 2-convex conditional lower prevision.
Then,

- $\underline{P}$ does not necessarily satisfy positive homogeneity, nor the condition $\underline{P}(X|B) \in [\inf X|B, \sup X|B]\ \forall X|B \in \mathcal{D}$ (*internality*).
- Non-internality cannot be two-sided.
- Centered 2-convex conditional previsions satisfy internality, have a 2-convex natural extension and agree with the Goodman-Nguyen relation ($=$ conditional implication / inclusion).

Properties of 2-coherent previsions

Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be 2-coherent.
Conjugate upper prevision $\overline{P}$:

$$\overline{P}(X|B) = -\underline{P}(-X|B), \forall X|B \text{ s.t. } -X|B \in \mathcal{D}$$

. Additional properties with respect to centered 2-convexity:

- $\underline{P}(X|B) \leq \overline{P}(X|B)\ \forall X|B \in \mathcal{D} : -X|B \in \mathcal{D}$.
- $\underline{P}$ is positively homogeneous.
- $\underline{P}$ has a 2-coherent natural extension.

The structured set $\mathcal{D}_{LIN}$

Let $\mathcal{X}$ be a linear space of gambles, $\mathcal{B} \subset \mathcal{X}$ the set of all events. Let also

- $1 \in \mathcal{B}$,
- $BX \in \mathcal{X}, \forall B \in \mathcal{B}, \forall X \in \mathcal{X}$,
- $\mathcal{B}^{\varnothing} = \mathcal{B} - \{\varnothing\}$.

Define

$$\mathcal{D}_{LIN} = \{X|B : X \in \mathcal{X}, B \in \mathcal{B}^{\varnothing}\}.$$

Coherence, convexity, 2-coherence and 2-convexity can be characterised on $\mathcal{D}_{LIN}$ through *sets of axioms*.

Some axioms for lower previsions

(DI) $\underline{P}(X|B) - \underline{P}(Y|B) \leq \sup\{X - Y|B\}$, $\forall X|B, Y|B \in \mathcal{D}_{LIN}$. (*Difference Internality*.)

(GBR) $\underline{P}(A(X - \underline{P}(X|A \wedge B))|B) = 0, \forall X \in \mathcal{X}, \forall A, B \in \mathcal{B}^{\varnothing}$: $A \wedge B \neq \varnothing$. (*Generalised Bayes Rule*.)

(PH) $\underline{P}(\lambda X|B) = \lambda \underline{P}(X|B), \forall X|B \in \mathcal{D}_{LIN}, \forall \lambda \geq 0$. (*Positive Homogeneity*.)

(NWH) $\underline{P}(\lambda X|B) \leq \lambda \underline{P}(X|B)$, $\forall \lambda < 0$ (*Negative Weak Homogeneity*.)

Characterisation through axioms

On $\mathcal{D}_{LIN}$,

- $\underline{P}$ is 2-*convex* iff (DI) and (GBR) hold.
- $\underline{P}$ is 2-*coherent* iff (DI), (GBR), (PH) and (NWH) hold.

Remark: *n-convex (n-coherent)* lower previsions $(n \geq 3)$ either are convex (coherent) themselves or have no *n*-convex (*n*-coherent) natural extension on any set.

A desirability approach

Given $\mathcal{D}_{LIN}$, define

$$\mathcal{X}^{\succeq} = \{X \in \mathcal{X} : \inf X \geq 0\},$$
$$\mathcal{X}^{\preceq} = \{X \in \mathcal{X} : \sup X \leq 0\},$$

and, $\forall B \in \mathcal{B}$,

$$\mathcal{R}(B) = \{X \in \mathcal{X} : BX = X\},$$
$$\mathcal{R}(B)^{\succ} = \{X \in \mathcal{R}(B) : \inf\{X|B\} > 0\},$$
$$\mathcal{R}(B)^{\prec} = \{X \in \mathcal{R}(B) : \sup\{X|B\} < 0\}.$$

Conditional coherence has been characterised by means of desirability axioms by Williams (1975, 2007).

2-coherence/convexity and desirability

Define, $\forall X|B \in \mathcal{D}_{LIN}$,

$$\underline{P}(X|B) = \sup\{x \in \mathbb{R} : B(X - x) \in \mathcal{A}\}.$$

where $\mathcal{A} \subseteq \mathcal{X}$ is a set of acceptable gambles.

a) $\lambda\mathcal{A} + \mathcal{R}(B)^{\succ} \subseteq \mathcal{A}, \forall\lambda \geq 0, B \in \mathcal{B}$;
b) $\mathcal{R}(B)^{\prec} \cap \mathcal{A} = \emptyset, \forall B \in \mathcal{B}$;
c) $(\mathcal{R}(B_1) \cap \mathcal{A}) + (\mathcal{R}(B_2) \cap \mathcal{A}) \subseteq \mathcal{R}(B_1 \vee B_2) \setminus \mathcal{R}(B_1 \vee B_2)^{\prec}$, $\forall B_1, B_2 \in \mathcal{B}$.

$\Rightarrow \underline{P}$ is 2-coherent on $\mathcal{D}_{LIN}$.

a') $\mathcal{A} + \mathcal{R}(B)^{\succ} \subseteq \mathcal{A}, \forall B \in \mathcal{B}$;
b) $\mathcal{R}(B)^{\prec} \cap \mathcal{A} = \emptyset, \forall B \in \mathcal{B}$.

$\Rightarrow$ $\underline{P}$ is 2-convex on $\mathcal{D}_{LIN}$;
$\underline{P}$ is centered iff $\mathcal{R}(B)^{\succ} \subseteq \mathcal{A}, \forall B \in \mathcal{B}$.

Comments

- a) and a') replace the cone conditions in (Williams, 1975)
- b) represents a condition of avoiding partial loss
- c) means that:

$$X_i \in \mathcal{A}, B_i X_i = X_i \ (i = 1, 2) \Rightarrow \sup(X_1 + X_2 | B_1 \vee B_2) \geq 0$$

$\Rightarrow$ $X_1 + X_2$ may be not accepted, but is not necessarily discarded (by b) with $B = B_1 \vee B_2$)

- If $\underline{P}$ is not centered, some $X|B$ s.t. $\inf(X|B) > 0$ might be not acceptable!
- It is possible to start from $\underline{P}$, 2-coherent or 2-convex, and define a set of acceptable gambles $\mathcal{A}'$ with suitable properties (details in the poster session).

2-convex and 2-coherent models - 1

- $\boldsymbol{P}$: finite partition; $\mathcal{L}$: linear space of random variables
- (Normalized) *capacity*: a mapping $c : 2^{\boldsymbol{P}} \to [0,1]$ s.t. $c(\emptyset) = 0, c(\Omega) = 1$ and, $\forall A_1, A_2 \in 2^{\boldsymbol{P}}$, if $A_1 \Rightarrow A_2$ then $c(A_1) \leq c(A_2)$.
- *Niveloid* (Dolecki, Greco, 1995): a functional $N : \mathcal{L} \to \overline{\mathbb{R}}$ s.t.

$$N(X+\mu) = N(X) + \mu, \forall X \in \mathcal{L}, \forall \mu \in \mathbb{R};$$
$$X \geq Y \Rightarrow N(X) \geq N(Y), \forall X, Y \in \mathcal{L}.$$

Niveloids are not necessarily centered.

2-convex and 2-coherent models - 2

- Proposition (Baroni, Pelessoni, Vicig, 2009)
  a) $\underline{P} : 2^{P} \to \mathbb{R}$ is a centered 2-convex lower prevision iff it is a capacity
  b) $P : \mathcal{L} \to \mathbb{R}$ ($\mathcal{L}$ linear space of gambles) is a 2-convex lower prevision iff it is a (finite-valued) niveloid.

$\Rightarrow$ 2-convex conditional lower previsions can define conditional capacities and niveloids.

- Using conjugate couples, like $(\underline{c}, \overline{c})$, we need 2-coherence to ensure $\underline{c} \leq \overline{c}$ (cf. also the case of bivariate p-boxes in (Pelessoni, Vicig, Montes, Miranda, submitted).)

Conclusions

- 2-coherent/convex lower previsions are very general uncertainty measures
- Often they may be too general to substitute coherence
- They are helpful in accomodating various uncertainty models in a unit betting/desirability scheme
- Further work needed on:
  - Extensions to unbounded gambles
  - Other properties
  - Incorporate additional uncertainty models ((conditional or not) risk measures,...)