Efficient $L1$-Based Probability Assessments Correction: Algorithms and Applications to Belief Merging and Revision

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Probability assessment

- A precise probability assessment is a quadruple $\pi = (V, U, p, \mathfrak{C})$, where
  - $V = \{X_1, \ldots, X_n\}$ is a finite set of propositional variables
  - U is a subset of V that contains the effective events taken into consideration
  - $p : U \rightarrow [0, 1]$ assigns a probability value to each variable in U
  - $\mathfrak{C}$ is a finite set of logical constraints which lie among all the variables in V

Coherence of probability assessment

- A precise probability assessment is coherent if there exists a probability distribution $\mu : 2^V \rightarrow [0,1]$ on the set of all truth-value assignment 2^V which satisfies the following properties
  1. for each $\alpha \in 2^V$, if there exists a constraint $c \in \mathfrak{C}$ such that $\alpha \not\models c$, then $\mu(\alpha) = 0$;
  2. $\sum_{\alpha \in 2^V} \mu(\alpha) = 1$;
  3. for each $X \in U$, $\sum_{\alpha \in 2^V, \alpha \models X} \mu(\alpha) = p(X)$.

Incoherence

- What to do if the probability assessment is not coherent ?
- A possible solution is to correct p in p' in a way that
  - $\pi' = (V, U, p', \mathfrak{C})$ is coherent
  - p' is as close as possible to p
- The correction is then a constrained minimization problem
- This approach follows the *principle of minimum change* of belief revision
- A distance between probability assessments is needed

L1 correction

- In this paper we use the L1 distance

$$d_1(p, p') = \sum_{i=1}^{n} |p(X_i) - p'(X_i)|$$

- L1-distance minimization has a simple interpretation, since it implies a direct minimal modification of each single probability value
- Moreover, the related correction procedure has a much lower computational cost than other distances
- Note that the correction is not unique, i.e. there can be infinitely many corrections for an incoherent assessment
- Anyway, all the corrections form a convex set $\mathcal{C}(\pi)$

Procedure Correct

- It is possible to convert the problem of checking the coherence of a probability assessment into a mixed integer programming (MIP) problem [Cozman]
- There exists fast procedures for solving MIP problems, even if this problem is NP-hard
- We shortly describe the procedure **Correct**
- The distance $\delta = d_1(p, p')$ between the original probability vector p and any of its corrections p' can be computed with a MIP program similar to the program for checking the coherence

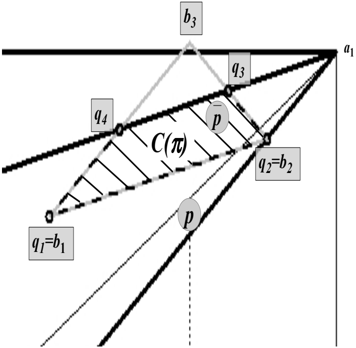
Procedure Correct

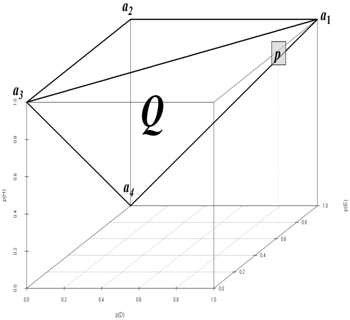
- If $\delta = 0$, p is already coherent and no correction is needed
- Otherwise, we want to find the extremal points $q_1, \ldots, q_s$ of $\mathcal{C}(\pi)$
- Indeed $\mathcal{C}(\pi) = \mathcal{Q} \cap \mathcal{B}_\pi(\delta)$ where
  - $\mathcal{Q}$ is the convex set (polytope) of all vectors q such that $(V, U, q, \mathfrak{C})$ is coherent
  - $\mathcal{B}_\pi(\delta)$ is the ball of all vectors q such that $d_1(p, q) \leq \delta$
- Fast procedures for face-enumeration and vertex-enumeration can be used to compute the result

Example

- We correct the following incoherent assessment with variables
  - $D \equiv$ "the athlete uses banned performance-enhancing drugs" (i.e. "doping")
  - $E \equiv$ "the athlete is showing a performance-enhancing in the last period"
  - $H \equiv$ "the athlete is showing a significant change in his/her biological profile"
- probability values $p(D) = 0.9$, $p(E) = 0.8$ and $p(H) = 0.9$
- logical constraint $\mathfrak{C} = \{E \vee H, \neg D \vee E, \neg D \vee H\}$

Example





Belief merging

- Given two coherent probability assessments $\pi_1 = (V, U, p, \mathfrak{C})$ and $\pi_2 = (V, W, q, \mathfrak{D})$, on the same propositional variables V, we want to find a probability assessment π_3 as fusion of π_1 and π_2
- The basic procedure is
  - Join together π_1 and π_2 in a incoherent probability assessment π'_3
  - Correct π'_3
- We propose two approaches to perform the first operation

Belief merging I

- The first approach is to compute a "weighted average" of π_1 and π_2 with weights ω and $1-\omega$
- We define $\pi_1 +_\omega \pi_2$ as the probability assessment $(V, U \cup W, r, \mathfrak{C} \cup \mathfrak{D})$, where $r : U \cup W \to [0,1]$ is now defined

$$r(x) = \begin{cases} p(x) & \text{if } x \in U \setminus W \\ q(x) & \text{if } x \in W \setminus U \\ \omega p(x) + (1-\omega) q(x) & \text{if } x \in U \cap W \end{cases}$$

- The merging operator is defined as

$$\pi_1 \oplus_\omega \pi_2 = \text{Correct}(\pi_1 +_\omega \pi_2)$$

Example

- Let $W = \{E, H, X_4 = (\neg D \wedge E \wedge H)\}$ and
- $\mathfrak{D} \equiv \mathfrak{C} \cup \{\neg D \vee \neg X_4, E \vee \neg X_4, H \vee \neg X_4\}$
- Let $\pi_1 = (V, W, \overline{p}, \mathfrak{D})$ with $\overline{p}(D) = 0.833, \overline{p}(E) = 0.867, \overline{p}(H) = 0.967$ and $\overline{p}(X_4) = 0$;
- Let $\pi_2 = (V, W, q, \mathfrak{D})$ with $q(E) = 0.867, q(H) = 0.967, q(X_4) = 0.01$
- Choosing $\omega = \frac{1}{2}$, we have the starting weighted assessment $\pi_1 +_{\frac{1}{2}} \pi_2$ with components V, $U \cup W = (D, E, H, X_4)$, $r = (0.8333, 0.8667, 0.9667, 0.005)$

Example

- $\pi_1 +_{\frac{1}{2}} \pi_2$ is incoherent with an $L1$ minimal distance $\delta = 0.01$
- The correction $\pi_1 \oplus_{\frac{1}{2}} \pi_2$ is the credal set with extremal values

$$
\begin{aligned}
q_1 &= (0.8333, 0.8742, 0.9667, 0.0075)\\
q_2 &= (0.8308, 0.8642, 0.9667, 0.00)\\
q_3 &= (0.8333, 0.8667, 0.9742, 0.0075)\\
q_4 &= (0.8308, 0.8667, 0.9642, 0.00)\\
q_5 &= (0.8358, 0.8692, 0.9667, 0.00)\\
q_6 &= (0.8258, 0.8667, 0.9667, 0.0075)
\end{aligned}
$$

Belief merging II

- A different approach is to create a probability assessment which maintains both numerical values
- The apparent contradiction is solved
  - by adding a new logical variable X_i', for each event $X_i \in U \cap W$ such that $p(X_i) \neq q(X_i)$, and
  - by assigning the values $r(X_i) = p(X_i)$ and $r(X_i') = q(X_i)$.
  - Moreover, the logical constraint $X_i = X_i'$ is added to $\mathfrak{C} \cup \mathfrak{D}$.
- $\pi_1 + \pi_2$ is obviously incoherent and the merging operation of π_1 and π_2 is computed as

$$\pi_1 \oplus_I \pi_2 = \mathrm{Correct}(\pi_1 + \pi_2).$$

Example

- As in the previous example, but we add a new event X'_4
- We start with the assessment $\pi_1 + \pi_2$ with components V, $U' = (D, E, H, X_4, X'_4)$, $r = (0.8333, 0.8667, 0.9667, 0.00, 0.01)$
- The logical constraints have also $\neg X_4 \vee X'_4, X_4 \vee \neg X'_4$
- The correction leads now to a precise assessment with numerical values $(0.8333, 0.8667, 0.9667, 0.00, 0.00)$

Comparison

- The main difference between the two approaches is that $\oplus_I$ tries to automatically solve the contradiction, while the operator $\oplus_\omega$ needs an explicit way of solving it.
- The approach of $\oplus_\omega$ is in some sense a supervised one, because the user must explicitly provide a weight ω,
- While $\oplus_I$ adopts an unsupervised approach, and these difference can leads to very different final results
- Thinking the probability assessments as belief states, the merging operators are a belief merging functions

Belief revision

- Suppose that $\pi_1 = (V, U, p, \mathfrak{C})$ represents our current belief state and a new reliable information $\pi_2 = (V, W, q, \mathfrak{D})$ arrives.
- We want to update our belief state with the new available information, with the idea that
  - we assume that the new information π_2 is correct
  - we allow to revise, as less as possible, π_1 in order to adapt it to the new information
- The revision can be performed as follows.
  - π_1 and π_2 are merged together with the operator $+_0$,
  - The resulting assessment is corrected by forbidding any change on the probabilities of the variables in W.
- The revision of π_1 with π_2 is then computed as

$$\pi_1 \star \pi_2 = Correct2(\pi_1 +_0 \pi_2, W)$$

Example

- Suppose we want to consider π_2 as valid
- We start with an initial assessment $\pi_1 +_0 \pi_2$ with components V, $U \cup W = (D, E, H, X_4)$, $W = (E, H, X_4)$, $r = (0.8333, 0.8667, 0.9667, 0.01)$ and logical constraints $\mathfrak{D}$
- The only possibility to correct it is to reduce the numerical evaluation $r(D) = 0.8333$ to $r'(D) = 0.823$
- Hence the revision $\pi_1 \star \pi_2$ is the precise assessment with components V , $U \cup W = (D, E, H, X_4)$, $r' = (0.8233, 0.8667, 0.9667, 0.01)$ and the same logical constraints $\mathfrak{D}$.

Comparison with Jeffrey's rule

- Revision operator $\star$ in general leads to an imprecise model
- It could be thought as an analogous of the famous Jeffrey's rule of combination
- The main difference is that $\star$ minimizes the probability mass dislocation from the original assessment, maintaining as much as possible the magnitude of the values, hence working in an "additive" way
- While Jeffrey's rule maintains as much as possible the odds ratios, hence working in a "multiplicative" way.
- Moreover the Jeffrey's rule produces a final probability assessment which could be too different from π since it inevitably alters all the values of p on $U \setminus W$
- While $\star$ tries to modify p as less as possible, in line with the belief revision methodology