Conformity and independence for coherent lower previsions

Enrique Miranda   Marco Zaffalon

University of Oviedo   IDSIA

ISIPTA '15

Research group

UNIMODE Research Unit

http://unimode.uniovi.es

Research interests:

- Imprecise probabilities: coherent lower previsions, p-boxes, random sets, independence.
- Fuzzy preference structures, intuitionistic fuzzy sets.
- Divergence measures.

Where we are

Where we are

Where we are

Introduction

Conformity of a marginal and a conditional lower prevision holds when they can be derived from some joint by means of natural extension.

We characterise conformity, also together with assumptions of epistemic irrelevance and independence.

In addition, we also give a behavioural characterisation of the strong product.

Outline

1. Preliminary concepts.
2. Conformity of marginal and conditional lower previsions.
3. Conformity and irrelevance.
4. Conformity and independence.
5. Conclusions and open problems.

Coherence and separate coherence

Consider two possibility spaces $\mathcal{X}_1, \mathcal{X}_2$, and let $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2) := \{f : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R} \text{ bounded}\}$.

A functional $\underline{P} : \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2) \to \mathbb{R}$ is a coherent lower prevision when it is the lower envelope of a family of expectations with respect to finitely additive probabilities.

Similarly, given $x_1 \in \mathcal{X}_1$, $\underline{P}(\cdot|x_1) : \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2) \to \mathbb{R}$ is a conditional coherent lower prevision when it is the lower envelope of a family of conditional expectations with respect to finitely additive probabilities.

In that case, $\underline{P}(\cdot|\mathcal{X}_1) := \sum_{x_1 \in \mathcal{X}_1} I_{x_1} \cdot \underline{P}(\cdot|x_1)$ is a separately coherent conditional lower prevision.

Marginalisation and conditioning

Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$.

Its restrictions to $\mathcal{X}_1, \mathcal{X}_2$-measurable gambles are its marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$.

For any $x_1 \in \mathcal{X}_1$, the conditional natural extension $\underline{E}(\cdot|x_1)$ is

$$\underline{E}(f|x_1) := \begin{cases} \sup\{\mu : \underline{P}(I_{x_1}(f-\mu)) \geq 0\} & \text{if } \underline{P}(x_1) > 0 \\ \inf_{x \in \mathcal{X}_2} f(x_1, x) & \text{otherwise.} \end{cases}$$

It represents the conditional behavioural assessments that can be derived from those modelled by $\underline{P}$.

Conformity of $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$

Given $\underline{P}_{\mathcal{X}_1}$ on $\mathcal{L}(\mathcal{X}_1)$ and $\underline{P}(\cdot|\mathcal{X}_1)$ on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$, we say that they are conforming when there exists $\underline{P}$ on $\mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$ with marginal $\underline{P}_{\mathcal{X}_1}$ and conditional natural extension $\underline{P}(\cdot|\mathcal{X}_1)$.

- $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$ are conforming $\iff$ $\underline{P}(\cdot|x_1)$ is vacuous whenever $\underline{P}_{\mathcal{X}_1}(x_1) = 0$.
- If $\mathcal{X}_1$ is finite and there is some $\underline{P}$ conforming with $\underline{P}_{\mathcal{X}_1}, \underline{P}(\cdot|\mathcal{X}_1)$, then the smallest one is $\underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$.

Conformity and irrelevance

$\underline{P}$ is said to model $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance when its conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$ satisfies epistemic irrelevance:

$$\underline{E}(f|x_1) = \underline{E}(f|x_1') \ \forall \mathcal{X}_2\text{-measurable } f, \forall x_1, x_1' \in \mathcal{X}_1.$$

$\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance when there exists $\underline{P}$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ and whose conditional natural extension $\underline{E}(\cdot|\mathcal{X}_1)$ satisfies

$$\underline{E}(f|x_1) = \underline{P}_{\mathcal{X}_2}(f(x_1, \cdot)) \ \forall f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2), \forall x_1 \in \mathcal{X}_1.$$

This is not equivalent to $\underline{P}$ modelling $\mathcal{X}_1$-$\mathcal{X}_2$ irrelevance and having marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$.

Set of conforming irrelevant joints

Let $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2})}$ be the set joints with marginals $\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2}$ and satisfying conformity with $\mathcal{X}_1$-$\mathcal{X}_2$-irrelevance.

- $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2})} \neq \emptyset \iff$ either $\underline{P}_{\mathcal{X}_1}(x_1) > 0\ \forall x_1$ or $\underline{P}_{\mathcal{X}_2}$ is vacuous.
- $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2})}$ is closed under lower envelopes.
- If $\mathcal{X}_1$ is finite and $\mathbb{P}^{irr}_{(\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2})} \neq \emptyset$, the smallest model in this set is $\underline{P}_{\mathcal{X}_1}(\underline{P}_{\mathcal{X}_2}) = \underline{P}_{\mathcal{X}_1}(\underline{P}(\cdot|\mathcal{X}_1))$, where $\underline{P}(\cdot|\mathcal{X}_1)$ is derived from $\underline{P}_{\mathcal{X}_2}$ by irrelevance.

Thus, conforming natural extension=irrelevant natural extension when $\mathcal{X}_1$ is finite (but not in general).

Conformity and independence

We say that $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ are conforming with $\mathcal{X}_1$-$\mathcal{X}_2$ independence when there exists $\underline{P}$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ satisfying

$$\underline{E}(f|x_1) = \underline{P}_{\mathcal{X}_2}(f(x_1,\cdot)) \ \forall f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2), x_1 \in \mathcal{X}_1$$
$$\underline{E}(f|x_2) = \underline{P}_{\mathcal{X}_1}(f(\cdot,x_2)) \ \forall f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2), x_2 \in \mathcal{X}_2.$$

Let $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2})}$ be the set these compatible joints.

- $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2})} \neq \emptyset \iff$ (a) either $\underline{P}_{\mathcal{X}_1}(x_1) > 0$ for every x_1 or $\underline{P}_{\mathcal{X}_2}$ is vacuous; and (b) either $\underline{P}_{\mathcal{X}_2}(x_2) > 0$ for every x_2 or $\underline{P}_{\mathcal{X}_1}$ is vacuous.

Connection with independence (I)

- If $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2})} \neq \emptyset$, then any independent product of $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$ belongs to $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2})}$.
- If $\mathcal{X}_1, \mathcal{X}_2$ are finite and $\mathbb{P}^{ind}_{(\underline{P}_{\mathcal{X}_1},\underline{P}_{\mathcal{X}_2})} \neq \emptyset$, then the smallest element of this set is the independent natural extension $\underline{P}_{\mathcal{X}_1} \otimes \underline{P}_{\mathcal{X}_2}$.

Consider $\underline{P}$ with marginals $\underline{P}_{\mathcal{X}_1}, \underline{P}_{\mathcal{X}_2}$, and the following condition:

$$\underline{P}(f) \leq \underline{P}(P_{\mathcal{X}_2}(f|\mathcal{X}_1)) \ \forall f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2), P_{\mathcal{X}_2} \geq \underline{P}_{\mathcal{X}_2}. \qquad (1)$$

- If $\mathcal{X}_2$ is finite, then $\underline{P} \leq \underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2} \iff$ it satisfies (1)...
- ... but it need not be an independent product.

Connection with independence (II)

For every $x_1 \in \mathcal{X}_1$ and $f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)$, let

$$\begin{aligned} f^{x_1} : \mathcal{X}_1 \times \mathcal{X}_2 &\to \mathbb{R} \\ (x_1', x_2) &\mapsto f(x_1, x_2). \end{aligned}$$

- If $\mathcal{X}_1, \mathcal{X}_2$ are finite, then $\underline{P}$ is an independent envelope $\iff$

$$\underline{P}(g - f) \geq \min_{x_1 \in \mathcal{X}_1} \underline{P}(g - f^{x_1}) \; \forall g, f \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_2)... \qquad (2)$$

- ...but not every prevision dominating $\underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$ satisfies (2).
- Thus, $\underline{P}_{\mathcal{X}_1} \boxtimes \underline{P}_{\mathcal{X}_2}$ is the only model satisfying (1) and (2).

Conclusions and open problems

Conclusions:

- Conformity clashes with the existence of zero lower probabilities.
- In the finite case the conforming natural extension becomes the irrelevant/independent natural extension.
- We have a behavioural characterisation of the strong product.

Open problems:

- Study this problem with other updating rules, like regular extension.
- Extension to more than two models.
- Connection with sets of gambles.