A pointwise ergodic theorem for imprecise Markov chains

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My boon companions

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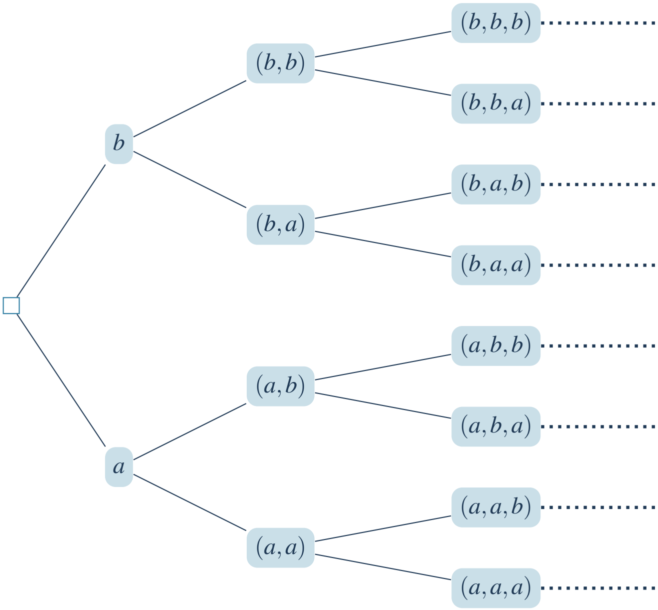
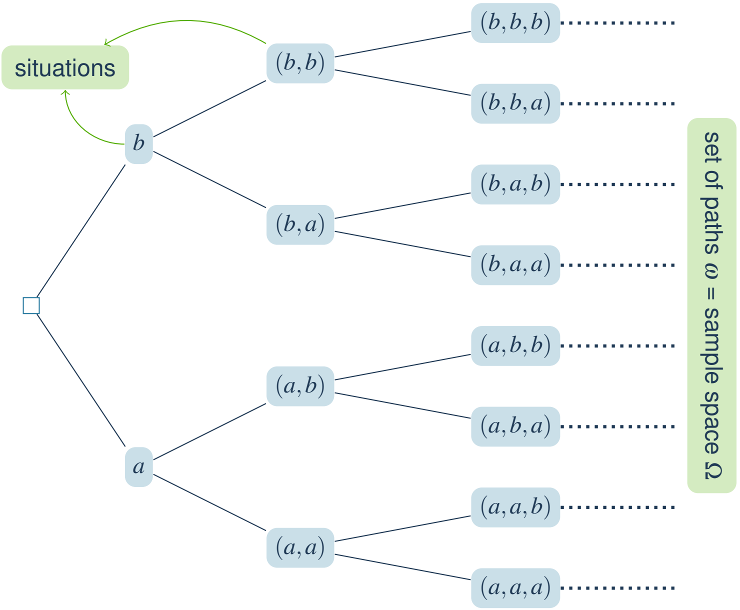
A discrete-time finite-state uncertain process

Uncertain variables $X_1, X_2, \ldots, X_n, \ldots$ assuming values in some finite set of states $\mathscr{X}$.

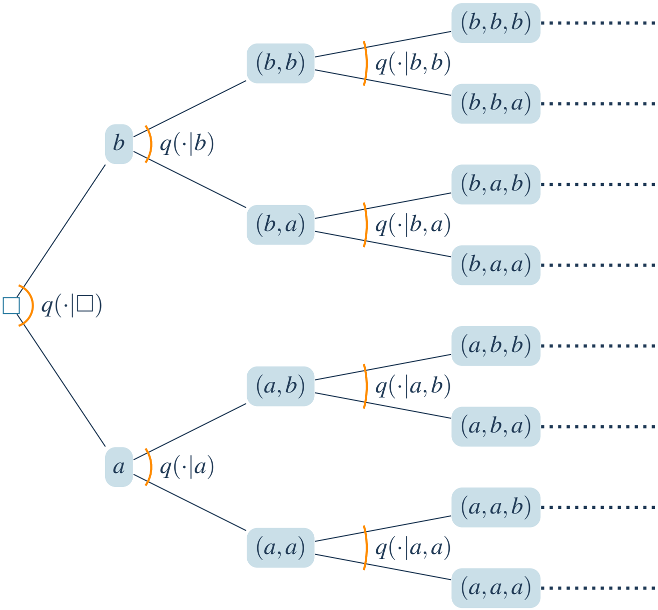
A simple discrete-time finite-state uncertain process



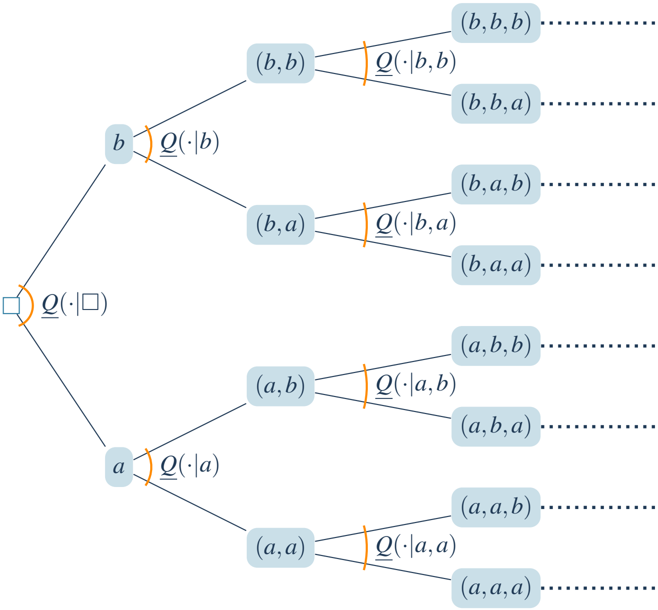
A simple discrete-time finite-state uncertain process



A simple discrete-time finite-state uncertain process



A simple discrete-time finite-state uncertain process



An event tree and its situations and paths

Situations are nodes in the event tree.

$$\text{situation } s = (x_1, x_2, \ldots, x_n) = x_{1:n}$$

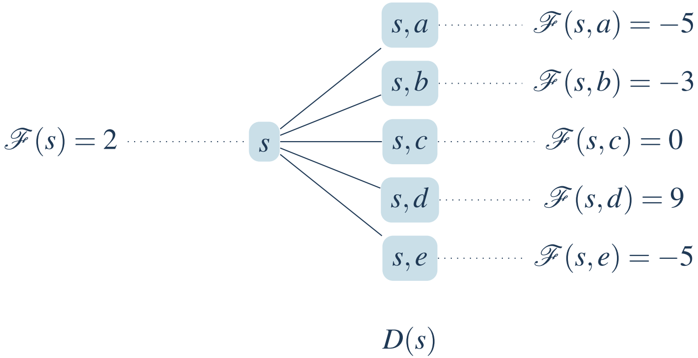
The sample space Ω is the set of all paths.

$$\text{path } \omega = (x_1, x_2, \ldots, x_n, \ldots) \in \mathscr{X}^{\mathbb{N}}$$

An event A is a subset of the sample space Ω: $A \subseteq \Omega$.

Processes and process differences

A real process $\mathscr{F}$ is a real function defined on situations:



and its process difference:

$$\Delta\mathscr{F}(s) = \mathscr{F}(s\,\cdot) - \mathscr{F}(s) \in \mathscr{G}(D(s)) \text{ for all situations } s$$

Sub- and supermartingales

We can use the local models $\underline{Q}(\cdot|s)$ to define sub- and supermartingales:

A submartingale $\underline{\mathscr{M}}$

is a real process such that in all non-terminal situations s:

$$\underline{Q}(\Delta\underline{\mathscr{M}}(s)|s) \geq 0.$$

A supermartingale $\overline{\mathscr{M}}$

is a real process such that in all non-terminal situations s:

$$\overline{Q}(\Delta\overline{\mathscr{M}}(s)|s) \leq 0.$$

Lower and upper expectations

The most conservative coherent lower and upper expectations on $\mathscr{G}(\Omega)$ that coincide with the local models and satisfy a number of additional continuity criteria (cut conglomerability and cut continuity):

Conditional lower expectations:

$$\underline{E}(f|s) := \sup\left\{\underline{\mathscr{M}}(s) : \limsup \underline{\mathscr{M}}(s\bullet) \leq f(s\bullet)\right\}$$

Conditional upper expectations:

$$\overline{E}(f|s) := \inf\left\{\overline{\mathscr{M}}(s) : \liminf \overline{\mathscr{M}}(s\bullet) \geq f(s\bullet)\right\}$$

Test supermartingales and strictly null events

A test supermartingale $\mathscr{T}$

is a non-negative supermartingale with $\mathscr{T}(\Box) = 1$.
(Very close to Ville's definition of a martingale.)

An event A is strictly null

if there is some test supermartingale $\mathscr{T}$ that converges to $+\infty$ on A:

$$\lim \mathscr{T}(\omega) = \lim_{n\to\infty} \mathscr{T}(\omega^n) = +\infty \text{ for all } \omega \in A.$$

If A is strictly null then

$$\overline{P}(A) = \overline{E}(\mathbb{I}_A) = \inf\left\{\overline{\mathscr{M}}(\Box) \colon \liminf \overline{\mathscr{M}} \geq \mathbb{I}_A\right\} = 0.$$

SLLN for submartingale differences (De Cooman and De Bock, 2013)

Consider any submartingale $\underline{\mathscr{M}}$ such that its difference process

$$\Delta\underline{\mathscr{M}}(s) = \underline{\mathscr{M}}(s\,\cdot) - \underline{\mathscr{M}}(s) \in \mathscr{G}(D(s)) \text{ for all non-terminal } s$$

is uniformly bounded. Then $\liminf \langle \underline{\mathscr{M}} \rangle \geq 0$ strictly almost surely, or in other words

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \Delta\underline{\mathscr{M}}(X_1,\ldots,X_{k-1})(X_k) = \liminf_{n\to+\infty} \frac{1}{n} \left[\underline{\mathscr{M}}(X_1,\ldots X_n) - \underline{\mathscr{M}}(\square) \right] \geq 0$$

In particular, for any real function f on $\mathscr{X}$:

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \left[f(X_k) - \underline{Q}(f(X_k|X_{1:k-1}) \right] \geq 0 \text{ strictly almost surely}$$

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In particular, for any real function f on $\mathscr{X}$:

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq \underline{Q}(f) \text{ strictly almost surely}$$

Imprecise Markov chains

Probability in the Engineering and Informational Sciences, **23**, 2009, 597–635. Printed in the U.S.A.
doi:10.1017/S0269964809990039

IMPRECISE MARKOV CHAINS AND THEIR LIMIT BEHAVIOR

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When the initial and transition probabilities of a finite Markov chain in discrete time are not well known, we should perform a sensitivity analysis. This can be done by considering as basic uncertainty models the so-called *credal sets* that these probabilities are known or believed to belong to and by allowing the probabilities to vary over such sets. This leads to the definition of an *imprecise Markov chain*. We show that the time evolution of such a system can be studied very efficiently using so-called *lower* and *upper expectations*, which are equivalent mathematical representations of credal sets. We also study how the inferred credal set about the state at time n evolves as $n \to \infty$: under quite unrestrictive conditions, it converges to a uniquely invariant credal set, regardless of the credal set given for the initial state. This leads to a non-trivial generalization of the classical Perron–Frobenius theorem to imprecise Markov chains.

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@ARTICLE{cooman2009,
  author = {{d}e Cooman, Gert and Hermans, Filip and Quaegehebeur, Erik},
  title = {Imprecise {M}arkov chains and their limit behaviour},
  journal = {Probability in the Engineering and Informational Sciences},
  year = 2009,
  volume = 23,
  pages = {597--635},
  doi = {10.1017/S0269964809990039}
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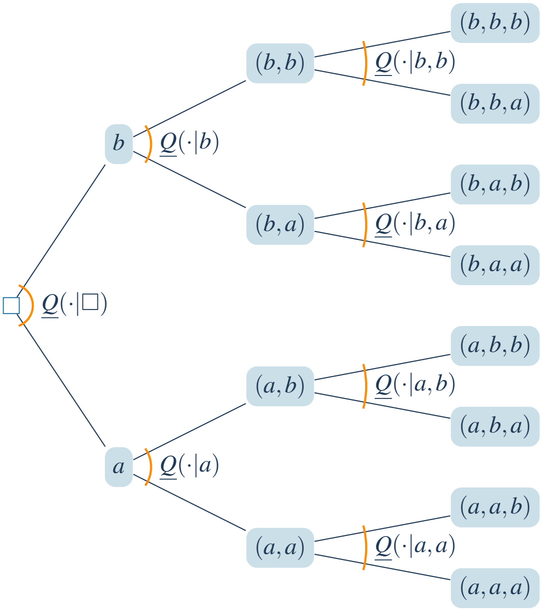
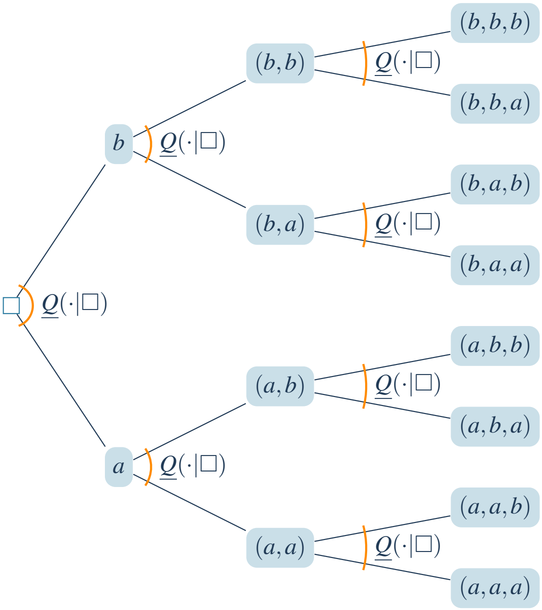
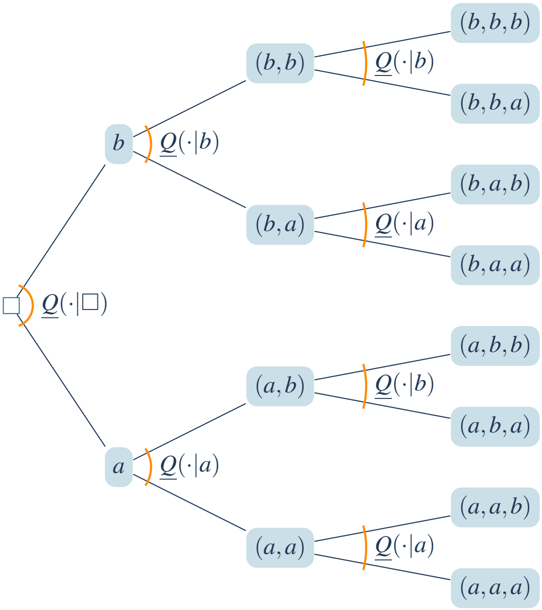
A simple discrete-time finite-state stochastic process



An imprecise IID model



An imprecise Markov chain



Stationarity and ergodicity

The lower expectation $\underline{E}_n$ for the state X_n at time n:

$$\underline{E}_n(f) = \underline{E}(f(X_n))$$

The imprecise Markov chain is Perron–Frobenius-like if for all marginal models $\underline{E}_1$ and all f:

$$\underline{E}_n(f) \to \underline{E}_\infty(f).$$

and if $\underline{E}_1 = \underline{E}_\infty$ then $\underline{E}_n = \underline{E}_\infty$, and the imprecise Markov chain is stationary.

In any Perron–Frobenius-like imprecise Markov chain:

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \underline{E}_n(f) = \underline{E}_\infty(f)$$

and

$$\underline{E}_\infty(f) \leq \liminf_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \leq \limsup_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \leq \overline{E}_\infty(f) \text{ str. almost surely.}$$

The essence of the argument

From

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \left[f(X_k) - \underline{Q}(f(X_k)|X_{k-1}) \right] \geq 0 \text{ strictly almost surely}$$

to

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \left[f(X_k) - \underline{E}_\infty(f) \right] \geq 0 \text{ strictly almost surely}$$