# Statistical Modelling under Epistemic Data Imprecision

## Some Results on Estimating Multinomial Distributions and Logistic Regression for Coarse Categorical Data

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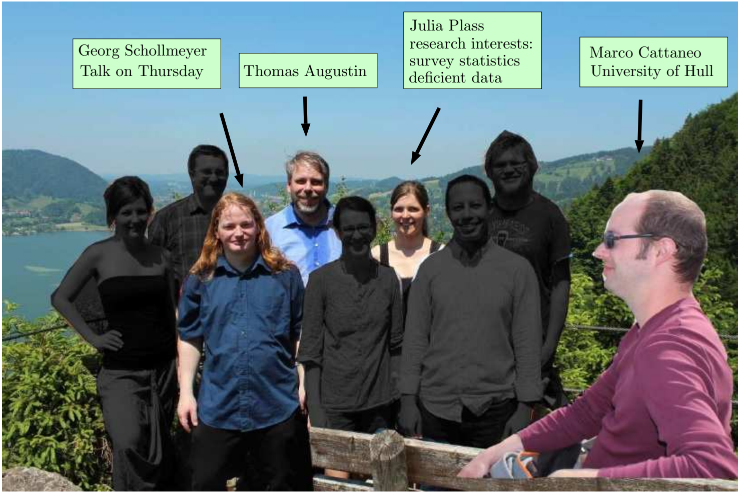
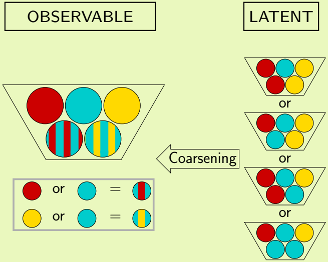


# Epistemic vs. ontic interpretation (Couso, Dubois, Sánchez, 2014)

## Epistemic imprecision:

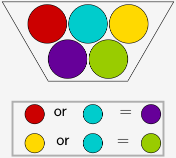
*"Imprecise observation of something precise"*




$\Rightarrow$ Truth is hidden due to the underlying coarsening mechanism

## Ontic imprecision:

*"Precise observation of something imprecise"*

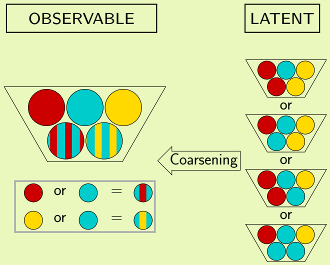



$\Rightarrow$ Truth is represented by coarse observation

# Examples of data under epistemic imprecision

## Epistemic imprecision:

*"Imprecise observation of something precise"*




$\Rightarrow$ Truth is hidden due to the underlying coarsening mechanism

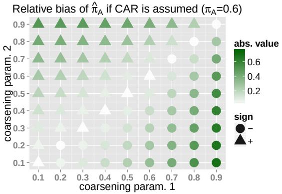
**Examples:**

- Matched data sets with partially overlapping variables
- Coarsening as anonymization technique
- Missing data as special case

**Here:** PASS-data
$\Omega_{\mathcal{Y}} = \{<, \geq, \text{na}\}$
"$<1000$", "$\geq 1000$" and
"$<1000$€ or $\geq 1000$€"(na)

# Already existing approaches

- Still common to **enforce precise results**
  $\Rightarrow$ Biased results:




- Variety of **set-valued approaches**
  - via random sets (e.g. Nguyen, 2006)
  - using Bayesian approaches (de Cooman, Zaffalon, 2004)
  - via likelihood-based belief function (Denœux, 2014)
  - via profile likelihood (Cattaneo, Wiencierz, 2012)

**Here:** Likelihood-based approach
influenced by methodology of partial identification (Manski, 2003)
coarse categorical data only

# Basic idea for the i.i.d. case (regression cf. poster)

OBSERVABLE

$\mathcal{Y}$ coarse data

$p_{\mathscr{y}_i} = P(\mathcal{Y}_i = \mathscr{y}_i),\ i = 1, \ldots, n$

Observation model $\mathcal{Q}$

error-freeness

coarsening mechanism

$q_{\mathscr{y}|y} = P(\mathcal{Y} = \mathscr{y} | Y = y)$

LATENT

Y latent variable

Main goal:

Estimation of $\pi_{ij} = P(Y_i = j)$

$\pi_{i1} = \pi_1, \ldots, \pi_{iK} = \pi_K$

$\boldsymbol{\gamma} = (\boldsymbol{q}_{\mathscr{y}|\boldsymbol{y}}^T,\ \boldsymbol{\pi}_{\boldsymbol{y}}^T)^T$

Use the **connection between $\boldsymbol{p}$ and $\boldsymbol{\gamma}$**

$$\Phi(\boldsymbol{\gamma}) = \mathbf{p}$$

Use random-set perspective and determine maximum-likelihood estimator $\hat{p}_{\mathscr{y}}$

Likelihood for parameters $\boldsymbol{p} = (p_1, \ldots, p_{|\Omega_{\mathcal{Y}}|-1})^T$

$L(\boldsymbol{p}) \propto \prod_{\mathscr{y} \in \Omega_{\mathcal{Y}}} p_{\mathscr{y}}^{n_{\mathscr{y}}}$ is *uniquely* maximized by

$\boxed{\hat{p}_{\mathscr{y}} = \frac{n_{\mathscr{y}}}{n},}$ $\quad \mathscr{y} \in \{1, \ldots, |\Omega_{\mathcal{Y}}| - 1\}$

and thus $\hat{p}_{|\Omega_{\mathcal{Y}}|} = 1 - \sum_{m=1}^{|\Omega_{\mathcal{Y}}|-1} \hat{p}_m$.

and the **invariance of the likelihood** under parameter transformations, i.e.:

$$\boxed{\hat{\Gamma} = \{\boldsymbol{\gamma} \mid \Phi(\boldsymbol{\gamma}) = \hat{\boldsymbol{p}}\}}$$

$\hat{\pi}_y \in \left[\frac{n_{\{y\}}}{n},\ \frac{\sum_{\mathscr{y} \ni y} n_{\mathscr{y}}}{n}\right]$

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**Illustration (PASS data)**

$n_{<} = 238,\ n_{\geq} = 835,\ n_{\mathrm{na}} = 338$

$\hat{\pi}_{<} \in \left[\frac{238}{1411},\ \frac{238+338}{1411}\right]$

# Reliable incorporation of auxiliary information

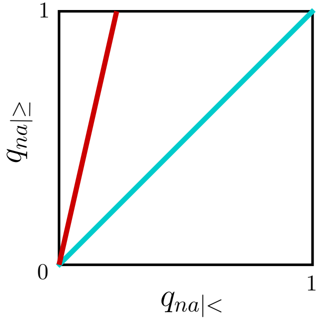
Starting from point-identifying assumptions, we use sensitivity parameters to allow inclusion of partial knowledge.

**Assumption about exact value of $R = \frac{q_{na|\geq}}{q_{na|<}}$** (Nordheim, 1984):

e.g. $\mathcal{Q}$ specified by R=1, R=4

where R=1 corresponds to CAR

(Heitjan, Rubin, 1991).



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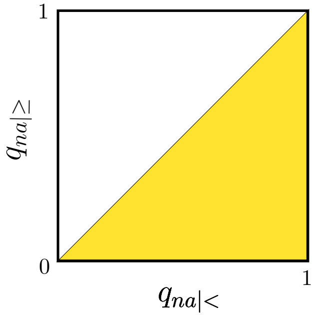
**Rough evaluation of $R$:**

e.g. $\mathcal{Q}$ specified by $R \leq 1$:

low income group has a higher tendency

to report "na"



# Summary and outlook

- Via the observation model $\mathcal{Q}$ maximum-likelihood estimators referring to the latent variable may be obtained for both cases
  - ... the homogeneous case
  - ... the case with categorical covariates (cf. poster)
- Proper inclusion of auxiliary information via further restrictions on $\mathcal{Q}$

**Next steps:**

- Inclusion of auxiliary information via sets of priors
- Likelihood-based hypothesis tests and uncertainty regions for coarse categorical data
- Consideration of other "deficiency" processes

# References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