On Two Composition Operator in Dempster-Shafer Theory

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Outline of the Lecture

What is a composition?

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Operators of Composition in Dempster-Shafer theory

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Conclusions - An Open Problem

What is a composition?

Composition assembles knowledge in the framework of uncertainty calculus.

- In probability theory: probability distributions.
- In possibility theory: possibility distributions.
- In Dempster-Shafer theory: basic probability assignments (commonality functions).
- In Valuation-Based systems: valuations.

Required properties of compositions

Let
- K and L are sets of variables;
- κ is a valuation for K, λ for L: $\kappa(K)$, $\lambda(L)$;

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- ... *(e.g. associativity under special conditions)*

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- composition is idempotent;
- composition is not used for knowledge updating;
- composition is different from combination, but it can be defined by combination $\oplus$ (and its reversal)

$$\kappa \triangleright \lambda = \kappa \oplus \lambda \ominus \lambda^{\downarrow K \cap L},$$

which prevents from the *double-counting* of knowledge.

Dempster's Operator of Composition

R. Jiroušek and P.P. Shenoy. Compositional models in valuation-based systems. IJAR, 2014(55), 1, 277–293.

Definition.
For commonality functions θ_1 on $\mathbb{X}_K$ and θ_2 on $\mathbb{X}_L$ ($K \neq \emptyset \neq L$) the commonality function of their *composition* $\theta_1 \overset{d}{\triangleright} \theta_2$ is defined for each nonempty $\mathbf{c} \subseteq \mathbb{X}_{K \cup L}$ by the formula:

$$(\theta_1 \overset{d}{\triangleright} \theta_2)(\mathbf{c}) = \begin{cases} \alpha^{-1} \frac{\theta_1(\mathbf{c}^{\downarrow K}) \cdot \theta_2(\mathbf{c}^{\downarrow L})}{\theta_2^{\downarrow K \cap L}(\mathbf{c}^{\downarrow K \cap L})} & \text{if } \theta_2^{\downarrow K \cap L}(\mathbf{c}^{\downarrow K \cap L}) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where α is a normalization constant defined as

$$\alpha = \sum_{\mathbf{d} \in 2^{\mathbb{X}_{K \cup L}} : \theta_2^{\downarrow K \cap L}(\mathbf{d}^{\downarrow K \cap L}) > 0} (-1)^{|\mathbf{d}|+1} \frac{\theta_1(\mathbf{d}^{\downarrow K}) \cdot \theta_2(\mathbf{d}^{\downarrow L})}{\theta_2^{\downarrow K \cap L}(\mathbf{d}^{\downarrow K \cap L})} .$$

Factorizing Operator of Composition

R. Jiroušek, J. Vejnarová and M. Daniel. Compositional models of belief functions. ISIPTA 2007, 243–252.

Definition.
For basic assignments μ_1 on $\mathbb{X}_K$ and μ_2 on $\mathbb{X}_L$ $(K \neq \emptyset \neq L)$ their *composition* $\mu_1 \,{}^{f}\!\triangleright\, \mu_2$ is defined for each nonempty $\mathbf{c} \subseteq \mathbb{X}_{K \cup L}$ by one of the following formulae:

(i) if $\mu_2^{\downarrow K \cap L}(\mathbf{c}^{\downarrow K \cap L}) > 0$ and $\mathbf{c} = \mathbf{c}^{\downarrow K} \bowtie \mathbf{c}^{\downarrow L}$ then

$$(\mu_1 \,{}^{f}\!\triangleright\, \mu_2)(\mathbf{c}) = \frac{\mu_1(\mathbf{c}^{\downarrow K}) \cdot \mu_2(\mathbf{c}^{\downarrow L})}{\mu_2^{\downarrow K \cap L}(\mathbf{c}^{\downarrow K \cap L})};$$

(ii) if $\mu_2^{\downarrow K \cap L}(\mathbf{c}^{\downarrow K \cap L}) = 0$ and $\mathbf{c} = \mathbf{c}^{\downarrow K} \times \mathbb{X}_{L \setminus K}$ then

$$(\mu_1 \,{}^{f}\!\triangleright\, \mu_2)(\mathbf{c}) = m_1(\mathbf{c}^{\downarrow K});$$

(iii) in all other cases, $(\mu_1 \,{}^{f}\!\triangleright\, \mu_2)(\mathbf{c}) = 0$.

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  - can be used for multidimensional model representation;
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- Both operators yield the same result if applied
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  - to Bayesian basic assignments.
- Both operators can be used to solve a marginal problem by the application of IPFP; for both of them it holds that if the process converges then the result is a joint extension of all the input basic assignments.

Properties of the Operators of Composition Differences

- IPFP: for factorizing operator Csiszár's convergence theorem holds true, it does not hold for Dempster operator (*R. Jiroušek, V. Kratochvíl. On Open Problems Connected with Application of the Iterative Proportional Fitting Procedure to Belief Functions. ISIPTA 2013*).

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- Computational complexity: Factorizing operator is much simpler than Demster operator.
  - factorizing operator $\sim \mathbf{c} \subset \mathbb{X}_{K \cup L} : \mathbf{c} = \mathbf{c}^{\downarrow K} \bowtie \mathbf{c}^{\downarrow L}$;
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- Computation of conditionals: for μ on $\mathbb{X}_M$ and $X_j, X_k \in M$,

$$\mu(X_k | X_j = \mathbf{a}) = \left(\nu_{X_j = \mathbf{a}} \,{}^{d}\!\triangleright\, \mu\right)^{\downarrow X_k},$$

where $\nu_{X_j=\mathbf{a}}$ is a one-dimensional bpa on $\mathbb{X}_j$ having just one focal element $\{\mathbf{a}\} \subset \mathbb{X}_j$, for which $\nu_{X_j=\mathbf{a}}(\{\mathbf{a}\}) = 1$.

Properties of the Operators of Composition

Conditional independence lemma

For commonality function $\theta(X, Y, Z)$ the conditional independence relation $X \perp\!\!\!\perp_\theta Y|Z$ holds true *iff*

$$\theta(X, Y, Z) = \theta(X, Z) \,{}^{d}\!\!\triangleright\, \pi(Y, Z).$$

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Factorization Lemma

For bpa $\mu(X,Y,Z)$ there exist functions

$$\phi : 2^{\mathbb{X}\times\mathbb{Z}} \longrightarrow \mathbb{R}^+, \qquad \psi : 2^{\mathbb{Y}\times\mathbb{Z}} \longrightarrow \mathbb{R}^+,$$

such that

$$\mu(\mathbf{a}) = \begin{cases} \phi(\mathbf{a}^{\downarrow\{X,Z\}}) \cdot \psi(\mathbf{a}^{\downarrow\{Y,Z\}}) & \text{if } \mathbf{a} = \mathbf{a}^{\downarrow\mathbb{X}\times\mathbb{Z}} \bowtie \mathbf{a}^{\downarrow\mathbb{Y}\times\mathbb{Z}} \\ 0 & \text{otherwise} \end{cases}$$

iff $\mu(X,Y,Z) = \mu(X,Z) \,{}^{f}\!\!\triangleright\, \mu(Y,Z)$.

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- In this case the local computations are possible in case that the following conjecture holds true.

Conjecture

Suppose μ_1, μ_2 and μ_3 are bpas on $\mathbb{X}_K$, $\mathbb{X}_L$, and $\mathbb{X}_M$, respectively.
If $L \supset (K \cap M)$ then,

$$(\mu_1 \,{}^{d}\!\triangleright\, \mu_2) \,{}^{f}\!\triangleright\, \mu_3 = \mu_1 \,{}^{d}\!\triangleright\, (\mu_2 \,{}^{f}\!\triangleright\, \mu_3).$$

Thank you for your attention