Comonotone lower probabilities for bivariate and discrete structures

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ISIPTA 2015, July 24

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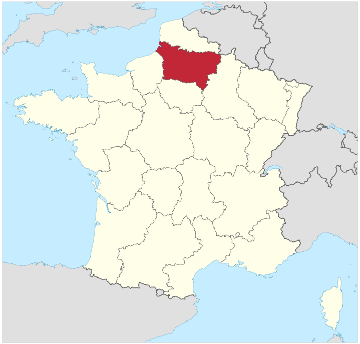

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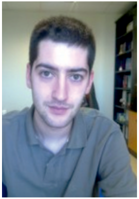
About the authors... I. Montes



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Overview

Overview

Motivation

- Multivariate distributions take into account the dependence between the variables.
- Possible dependence between the variables: independence, comonotonicity, countermonotonicity, ...
- What happens when we have imprecise information?
- Independence for imprecise probabilities has already been studied.
- And what about comonotonicity?

Overview

Comonotone random variables

Let (X,Y) be a random vector defined on $\mathcal{X} \times \mathcal{Y}$, and denote by $P_{\mathrm{X,Y}}$ its probability distribution. $P_{\mathrm{X,Y}}$ is called comonotone when the following equivalent conditions hold:

- For any $(x,y) \in \mathcal{X} \times \mathcal{Y}$, either $P(X \leq x, Y > y) = 0$ or $P(X > x, Y \leq y) = 0$.
- The support of $P_{\mathrm{X,Y}}$ is an increasing subset of $\mathbb{R}^2$.
- $F_{\mathrm{X,Y}}(x,y) = \min(F_{\mathrm{X}}(x), F_{\mathrm{Y}}(y))$ for any (x,y).

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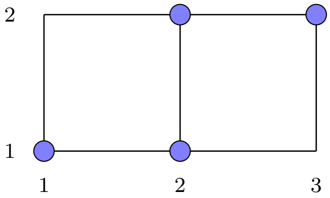
	ω_1	ω_2	ω_3	ω_4
X	2	2	3	1
Y	1	2	2	1



Comonotone lower probabilities

Assume that $\underline{P} : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to [0, 1]$ is a lower probability modelling the imprecise knowledge about $P_{\mathrm{X,Y}}$. We say that $\underline{P}$ is comonotone when any $P \in \mathcal{M}(\underline{P})$ is comonotone.

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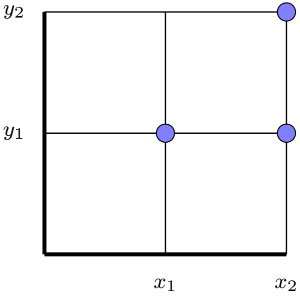
Example: Consider $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2\}$ ($x_1 < x_2$ and $y_1 < y_2$). If $\overline{P}(\{(x_1, y_2)\}) = 0$, any $P \in \mathcal{M}(\underline{P})$ is comonotone:



Characterizations

Comonotone lower probabilities

$\mathrm{supp}(\underline{P})$ is an increasing set

$\forall (x,y)$, either
$\overline{P}(\{(u,v) : u > x, v \leq y\}) = 0$ or
$\overline{P}(\{(u,v) : u \leq x, v > y\}) = 0$.

Any $P \in \mathcal{M}(\underline{P})$ is comonotone.

$$\underline{F}(x,y) = \min(\underline{F}_{\mathrm{X}}(x), \underline{F}_{\mathrm{Y}}(y))$$
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Overview

Introduction

Comonotonicity for lower probabilities

Building comonotone lower probabilities

Conclusions

Building comonotone lower probabilities

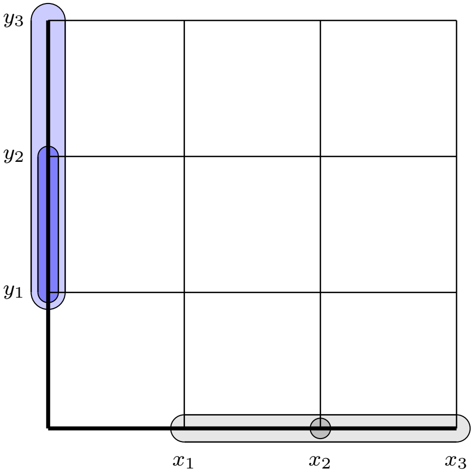
Theorem

Let Bel_{X} and Bel_{Y} be two belief function with nested focal elements. Then, it is always possible to build a joint coherent comonotone lower probability with these fixed marginals, and in fact, it is also a belief function with nested focal elements.

Theorem

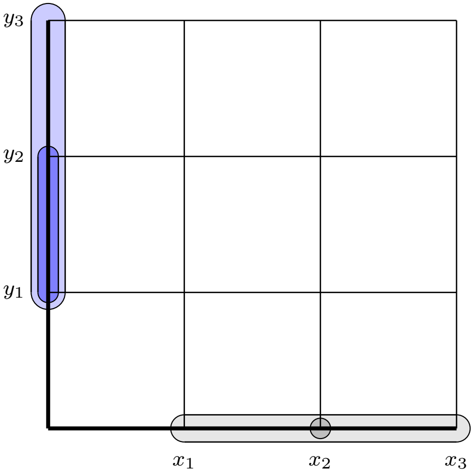
Let Bel_{X} and Bel_{Y} be two belief functions with focal sets $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ such that $A_i = [\underline{a}_i, \overline{a}_i]$ and $B_i = [\underline{b}_i, \overline{b}_i]$ are intervals for $i = 1, \ldots, n$ and $m_{\mathrm{X}}(A_i) = m_{\mathrm{Y}}(B_i)$. Then, there exists a joint coherent comonotone lower probability with these fixed marginals. Furthermore, it is a belief function.

Example



$$m_X(\{x_2\}) = 0.6, \qquad m_X(\{x_1, x_2, x_3\}) = 0.4.$$
$$m_Y(\{y_1, y_2\}) = 0.2, \quad m_Y(\{y_1, y_2, y_3\}) = 0.8.$$

Example



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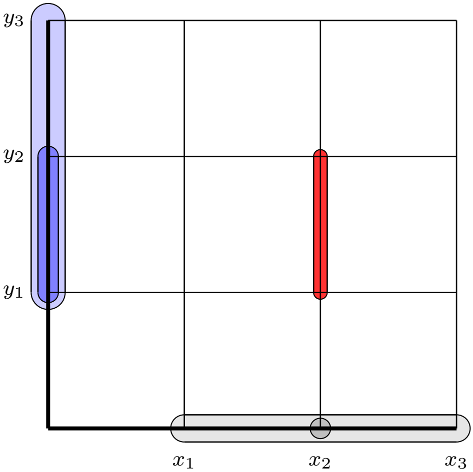
Example



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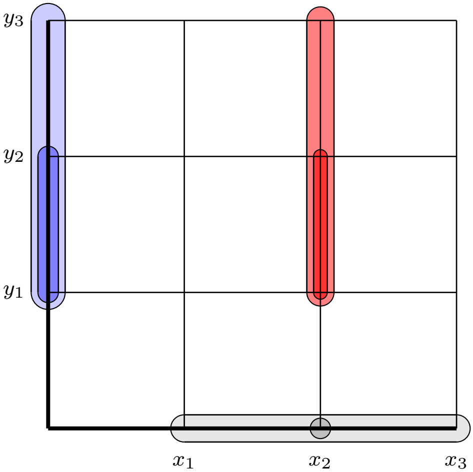

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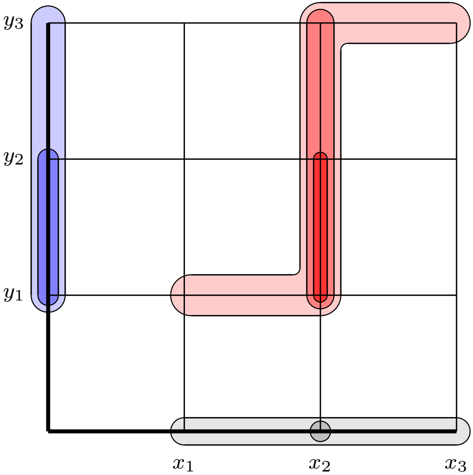

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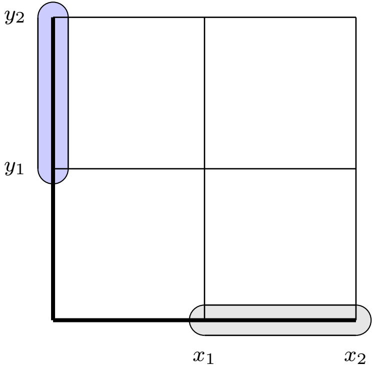
Non-uniqueness

The joint comonotone belief function may not be unique!!



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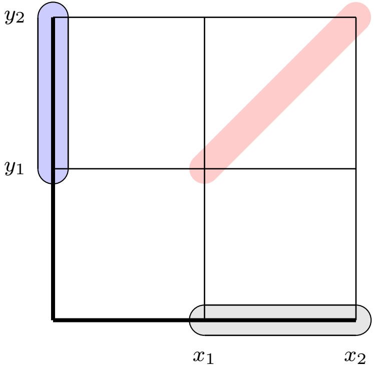

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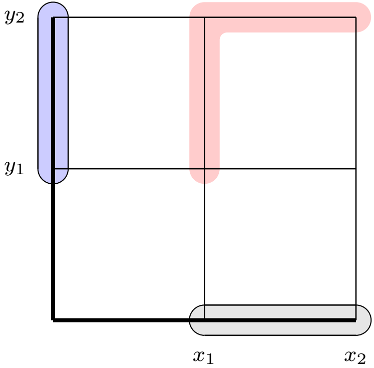

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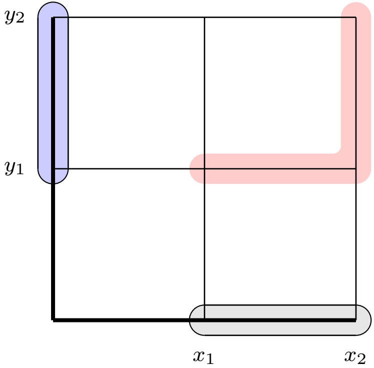

Overview

Highlights

This work investigates the notion of comonotonicity for lower probabilities.

- Definition, characterizations and properties.
- Not all marginals allow to define a joint comonotone coherent lower probability.
- It does for particular cases of belief functions.
- Comonotonicity for bivariate p-boxes: too much restrictive.
- All our results can be extended to countermonotonicity.

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