ISIPTA '15
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Fully Conglomerable Coherent Upper Conditional Prevision Defined by the Choquet Integral with respect to its Associated Hausdorff Outer Measure

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A new model of coherent upper conditional prevision based on Hausdorff outer measures

(Doria, 2012 Theorem 2)

Let m be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B. Then for each $B \in \boldsymbol{B}$ the functionals $\overline{P}(X|B)$ defined on $L(B)$ by

$$\overline{P}(X|B) = \begin{cases} \dfrac{1}{h^s(B)} \displaystyle\int_B X dh^s & if \quad 0 < h^s(B) < +\infty \\ m(XB) & if \quad h^s(B) = 0, +\infty \end{cases}$$

are separately coherent upper conditional previsions.

- The unconditional prevision is obtained when the conditioning event is Ω.
- $\overline{P}(X|\boldsymbol{B})(\omega)$ is the random variable equal to $\overline{P}(X|B)$ if $\omega \in B$.

Coherent upper conditional probabilities are obtained when only indicator functions of events are considered.

(Doria, 2012 Theorem 3)

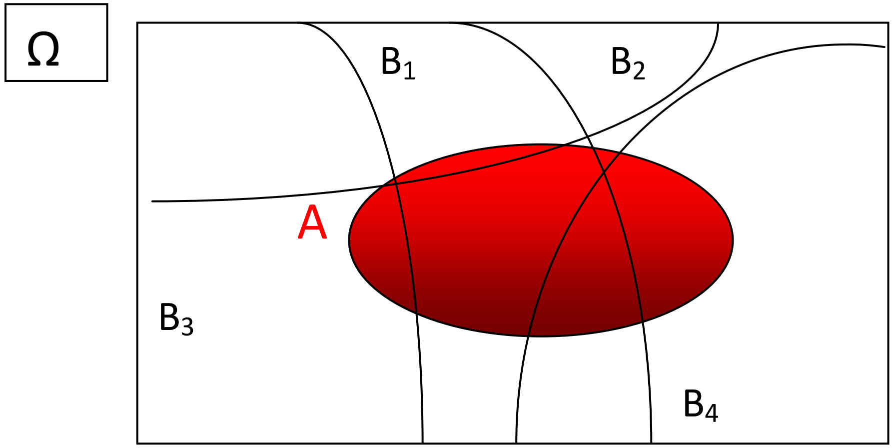
Let m be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B. Then for each $B \in \boldsymbol{B}$ the function $\overline{P}(\cdot\,|B)$ defined on $\wp(B)$ by

$$\overline{P}(A|B) = \begin{cases} \dfrac{h^s(AB)}{h^s(B)} & if \quad 0 < h^s(B) < +\infty \\ \\ m(AB) & if \quad h^s(B) = 0, +\infty \end{cases}$$

is a coherent upper conditional prevision.

Given a finite partition $\boldsymbol{B} = \{B_i\}_{i=1}^{n}$ of Ω and an additive probability P

Law of total probability



$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

Does a coherent upper prevision satisfy a similar law for every random variable X defined on Ω and for every **arbitrary** partition $\boldsymbol{B}$?

$$\overline{P}(X) = \sum_{B \in \boldsymbol{B}} \overline{P}(X|B)\overline{P}(B) = \overline{P}\left(\overline{P}(X|\boldsymbol{B})\right)$$

MOTIVATIONS

1) A new model of coherent upper conditional prevision based on Hausdorff outer measures has been introduced because conditional expectation defined in the axiomatic way by the Radon-Nikodym derivative may be fail to be coherent. So it is important to prove that the price of coherence is not to lose the **disintegrability property** that is a property satisfied by conditional expectation in the axiomatic approach.

2) In Walley **full conglomerability** is required as a rational axiom for coherent upper prevision since it assures that it can be extended to coherent conditional upper prevision for any partition $\boldsymbol{B}$ of $\boldsymbol{\Omega}$**.**

Conglomerability principle

If a random variable X is B-desirable, i.e. we have a disposition to accept X for every set B in the partition $\boldsymbol{B}$*, then X is desirable.*

Definition 1.

A coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ is ***disintegrable*** with respect to a partition $\boldsymbol{B}$ of Ω if the following equality holds for every bounded variable $X \in L(\Omega)$

$$\overline{P}(X) = \overline{P}(\overline{P}(X|\boldsymbol{B}))$$

(disintegration property)

Definition 2.

A coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ is ***conglomerable*** with respect to a partition $\boldsymbol{B}$ if the following implication holds for every bounded variable $X \in L(\Omega)$.

$$\overline{P}(X|\boldsymbol{B}) \geq 0 \Rightarrow \overline{\mathrm{P}}(\mathrm{X}) \geq 0$$

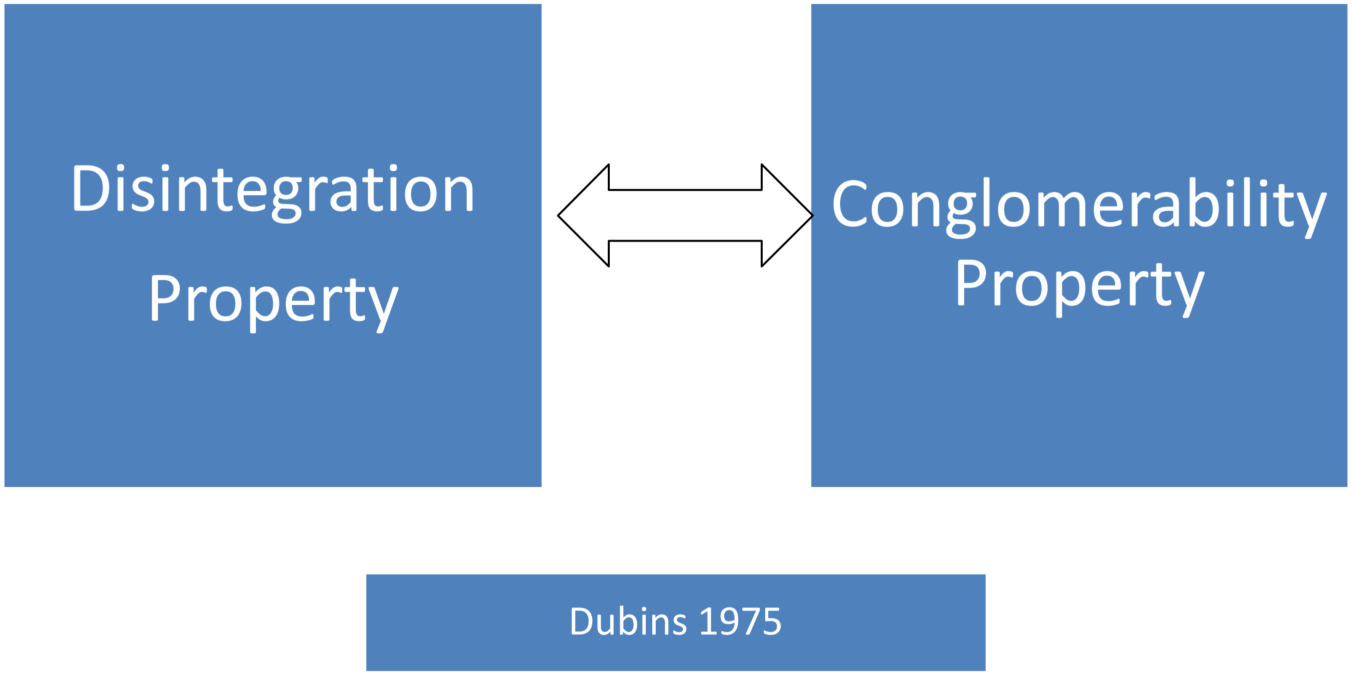
(conglomerability property)

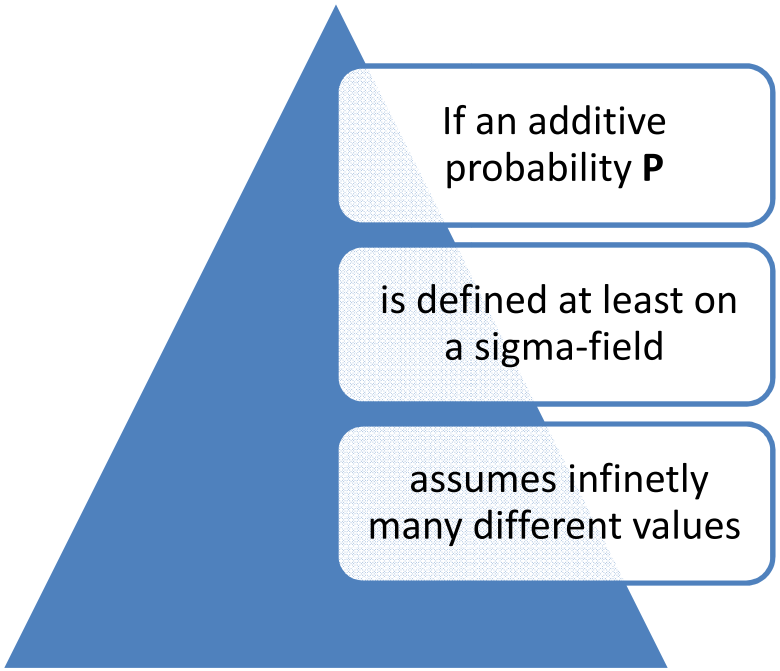
Definition 3.

A coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ is ***fully conglomerable*** if the following implication holds for every bounded variable $X \in L(\Omega)$ and for every partition $\boldsymbol{B}$

$$\overline{P}(X|\boldsymbol{B}) \geq 0 \Rightarrow \overline{\mathrm{P}}(\mathrm{X}) \geq 0 \quad \text{(full conglomerability)}$$

For **linear conditional previsions** $P(X|\boldsymbol{B})$ the problem has been investigated in literature.




If an additive probability **P**
is defined at least on a sigma-field
assumes infinetly many different values

P is fully conglomerable

⇔

P is countably additive

Schevish, Seidenfeld Kadane, 1984

Walley, 1991 section 6.9

For an arbitrary partition *B* countably additivity is not sufficient to assure that the conglomerability property is satisfied.

Kadane, Schervish and Seidenfeld, 1986

Examples of non-conglomerable linear previsions are given in **Walley (1991, sections 6.6.6,6.6.7)**

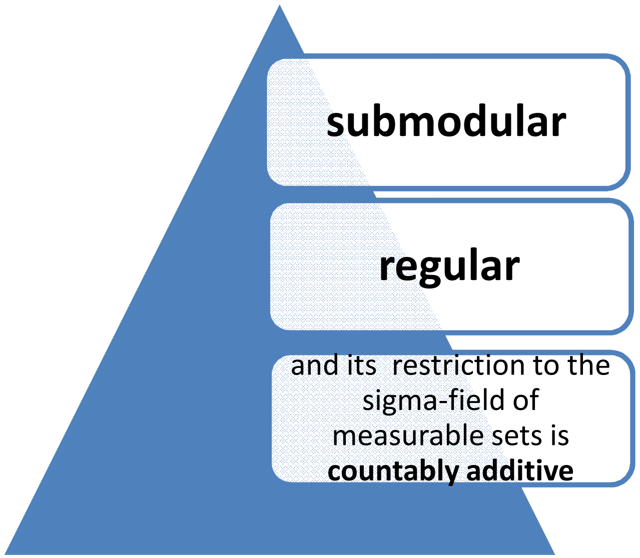
Consequences of failure of conglomerability are investigated in decision making where non-conglomerability of finitely additive probabilities leads to a violation of the decision-theoretic principle of admissibility as proven in **Kadane, Schervish and Seidenfeld (1986**).

Moreover failure of conglomerability has consequence in sequential decision problems **(Kadane, Schervish and Seidenfeld, 2008**).

Does the natural extension of a coherent countably additive probability satisfy the disintegration property and the conglomerability property on every arbitrary partition?

An affermative answer is given for coherent upper conditional previsions defined with respect to Hausdorff outer measures

The result is based on the fact that every t-dimensional Hausdorff outer measure is:



Main Results

Theorem 1. Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension t then the new model of coherent upper conditional prevision based on Hausdorff outer measures satisfies the **disintegration property**

$$\overline{P}(X) = \overline{P}(\overline{P}(X|\boldsymbol{B}))$$

for every random variable X defined on Ω and for every **arbitrary** partition **B**.

Sketch of the proof:

■ Since Hausdorff outer measures are **submodular** and every random variable X and every constant c are ***comonotonic***, we consider two comonotonic classes

$$\boldsymbol{C} = \left\{\overline{P}(X|\boldsymbol{B}), c\right\} \text{ and } \boldsymbol{C}' = \{X, c\}$$

so that, by Proposition 10.1 of Denneberg 1994 (based on the Hahn-Banach Theorem), there exist two additive set functions α and α' on $\wp(\Omega)$, which agree with h^t on the σ-field of the h^t-measurable sets, such that

$$\int \overline{P}(X|\boldsymbol{B})dh^t = \int \overline{P}(X|\boldsymbol{B})d\alpha'$$

$$\int X dh^t = \int X d\alpha$$

■ Every Hausdorff outer measure h^t is **regular,** that is for every set $A \in \wp(\Omega)$ there is a h^t-measurable set A' such that $h^t(A) = h^t(A')$.

■ Since Ω is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension t we have that the restriction of $\overline{P}(\cdot\,|B)$ to the class of h^t- measurable sets is a countably additive probability (μ_Ω).

So for every partition $\boldsymbol{B}$ there is at most a countable subclass $\boldsymbol{B}^*$ of $\boldsymbol{B}$ of sets B with positive coherent upper probability μ_Ω.

Thus for very random variable $X \in L(\Omega)$ the disintegration property is satisfied for every partition $\boldsymbol{B}$ since the following equalities hold:

$$\overline{P}\left(\overline{P}(X|\boldsymbol{B})\right) = \frac{1}{h^t(\Omega)}\int_{\Omega} \overline{P}(X|\boldsymbol{B})dh^t =$$

$$\frac{1}{h^t(\Omega)}\int_{\Omega} \overline{P}(X|\boldsymbol{B})d\alpha' =$$

$$\sum_{B\in\boldsymbol{B}^*}\left(\frac{1}{h^t(B)}\int_{B} Xd\alpha\right)\frac{h^t(B)}{h^t(\Omega)} =$$

$$\frac{1}{h^t(\Omega)}\sum_{B\in\boldsymbol{B}^*}\int_{B} Xd\alpha =$$

$$\frac{1}{h^t(\Omega)}\int_{\Omega} Xdh^t = \overline{P}(X)$$

For coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ defined by Hausdorff outer measure

disintegrability with respect to every arbitrary partition ***B***

$\Rightarrow$

full conglomerability

$\Leftarrow$?

Theorem 2. Let Ω be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension t then the new model of coherent upper conditional prevision based on Hausdorff outer measures satisfies the **conglomerability property**

$$\overline{P}(X|\boldsymbol{B}) \geq 0 \Rightarrow \overline{P}(X) \geq 0$$

for every random variable X defined on Ω and for every **arbitrary** partition $\boldsymbol{B}$.

Sketch of the proof:

By the coherence of the upper unconditional prevision $\overline{P}$ we have that for the random variable $\overline{P}(X|\boldsymbol{B})$ the following implication holds

$$\overline{P}(X|\boldsymbol{B}) \geq 0 \Rightarrow \overline{P}(\overline{P}(X|\boldsymbol{B})) \geq 0.$$

Since the random variable $\overline{P}(X|\boldsymbol{B})$ satisfies the disintegration property

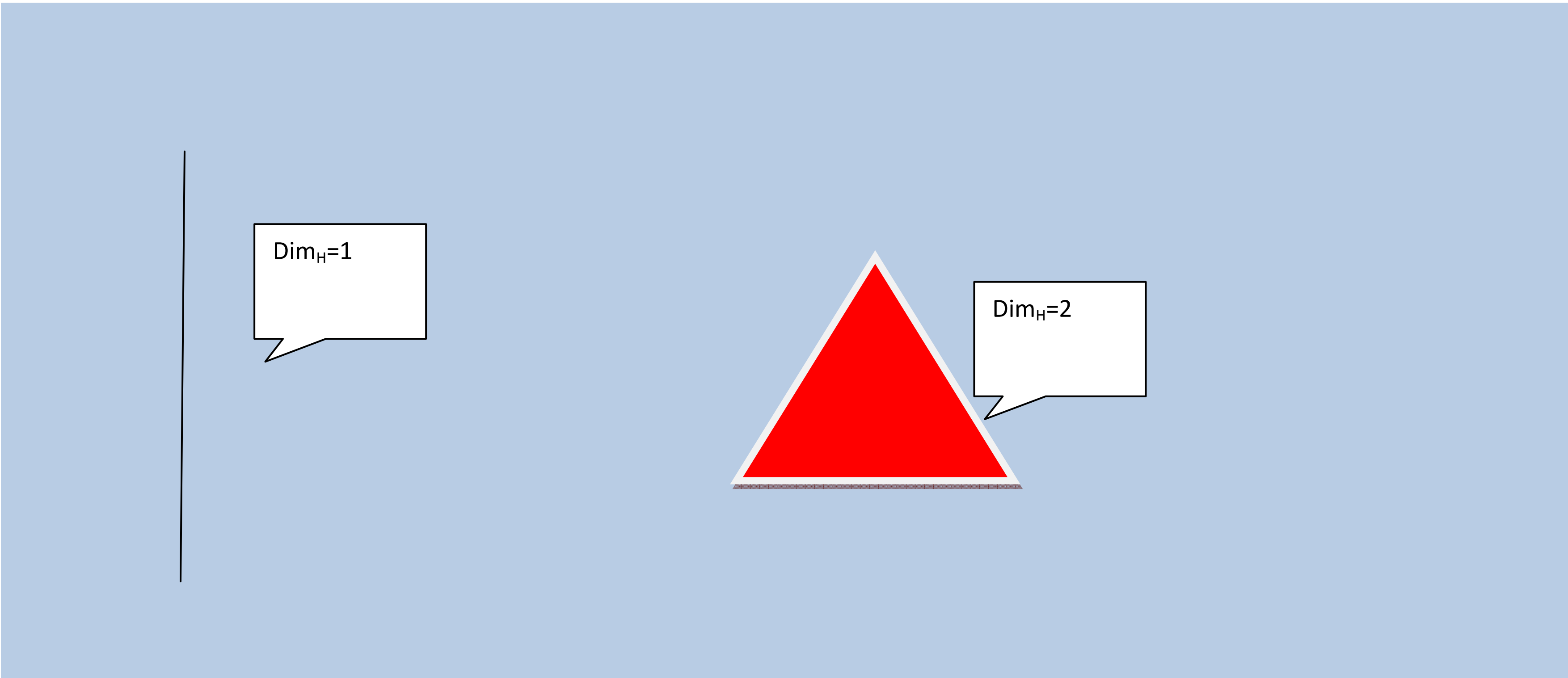
($\overline{P}(X) = \overline{P}(\overline{P}(X|\boldsymbol{B}))$) for *every partition* $\boldsymbol{B}$ thus we obtain

$$\overline{P}(X|\boldsymbol{B}) \geq 0 \Rightarrow \overline{P}(X) \geq 0$$

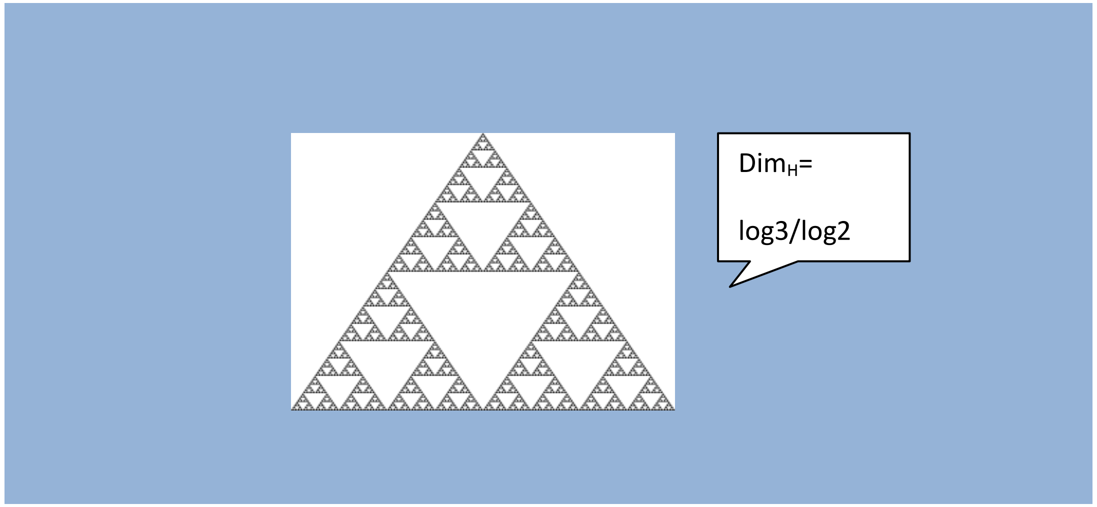
that is $\overline{P}(X|\boldsymbol{B})$ is **fully conglomerable**.

Hausdorff dimension of a set

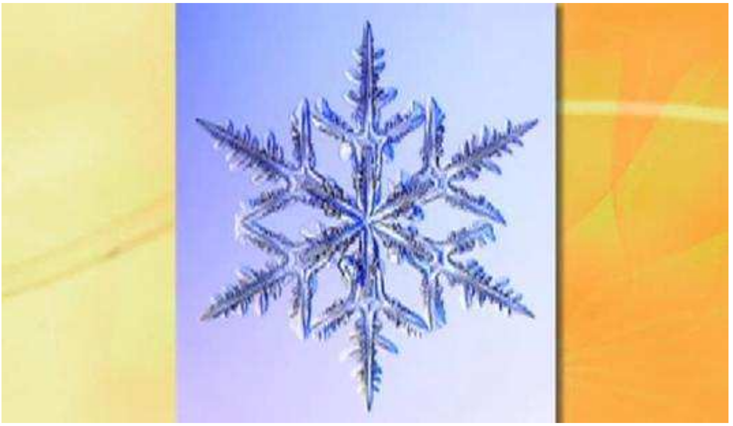
Ω



Sierpinsky Triangle



Self-similar objects in nature

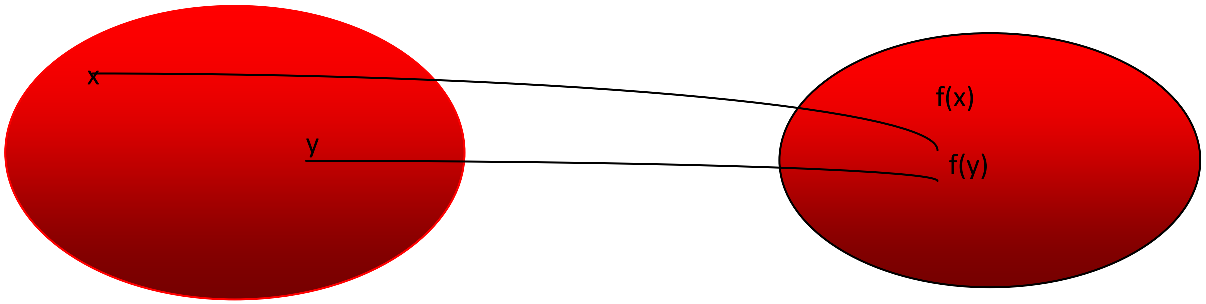


A set with non-integer Hausdorff dimension can be built as **attractor** of a finite set of contractions (Iterated Function System IFS, Barnsley, 1988)

A **contraction** is a function f on a metric space (Ω, d) such that $\forall\ x, y\ \in \Omega$

$$d\big(f(x), f(y)\big) \leq r\ d(x, y)$$

where $0 < r < 1$ is a constant.



A contraction that transforms every subset of Ω to a geometrically similar set is called a **similitude**.

For a finite set of contractions $\{f_1, f_2, \dots, f_m\}$ there exists a unique non-empty set compact invariant set K (Falconer 1986) called attractor.

$$K\ invariant \Leftrightarrow K = \bigcup_{i=1}^{m} f_i\,(K)$$

If the Open Set Condition holds then the compact invariant set K is **self-similar** and the **Hausdorff dimension and the similarity dimension are equal .**

The set is **self-similar** if the whole set is composed of smaller parts which are geometrically similar to whole set.

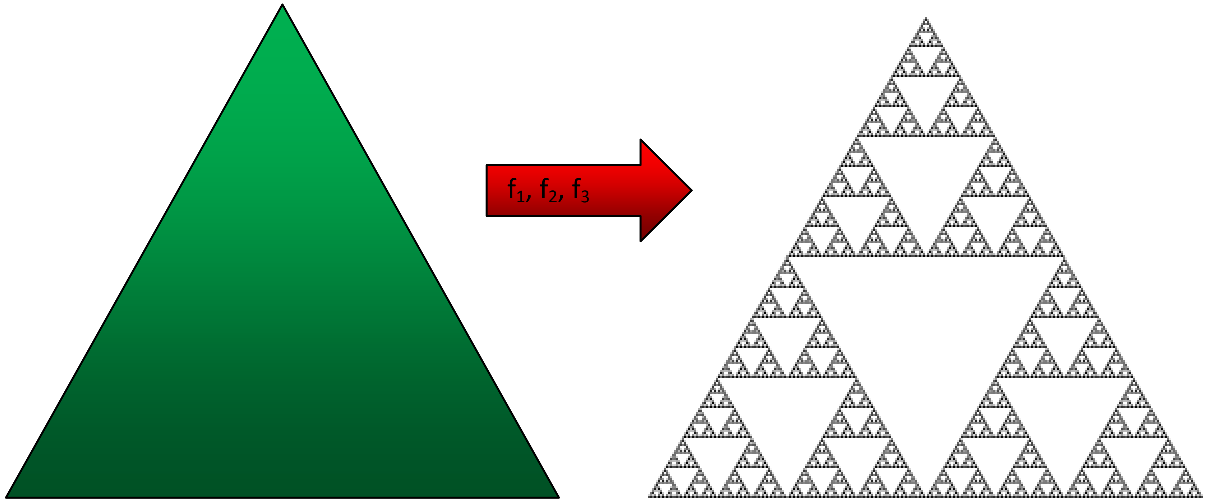
For a finite set of similitudes with similarity ratios $r_i\ i = 1, \dots m$ the **similarity dimension** is the unique positive number **s** for which

$$\sum_{i=1}^{m} r_i^s = 1$$

Example

The Sierpinsky Triangle is the attractor of the following set of similitudes

$$f_1 = \begin{cases} X = \frac{1}{2}x \\ Y = \frac{1}{2}y \end{cases} \quad f_2 = \begin{cases} X = \frac{1}{2}x + 1 \\ Y = \frac{1}{2}y \end{cases} \quad f_3 = \begin{cases} X = \frac{1}{2}x + \frac{1}{4} \\ Y = \frac{1}{2}y + \frac{\sqrt{3}}{4} \end{cases}$$



$$1 = \sum_{i=1}^{3} \left(\frac{1}{2}\right)^{s} \Rightarrow s = \frac{log3}{log2} = dim_H$$